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Point spectrum of the operator matrices with the Fredholm integral operators

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In mathematics, Fredholm operators are certain operators that arise in the Fredholm theory of integral equations. They are named in honour of Erik Ivar Fredholm.

A linear operator $\mathcal{A}$ from a Banach space X to a Banach space Y is called a Fredholm operator if
1. $\mathcal{A}$ is closed;
2. the domain $D(\mathcal{A})$ of $\mathcal{A}$ is dense in X;
3. $\alpha(\mathcal{A})$, the dimension of the null space $N(\mathcal{A})$ of $\mathcal{A}$, is finite;
4. $R(\mathcal{A})$, the range of $\mathcal{A}$, is closed in Y;
5. $\beta(\mathcal{A})$, the codimension of $R(\mathcal{A})$ in Y, is finite.

In particular, on the spaces $C[a;b]$ or $L_2[a;b]$ an operator of the form

$$(\mathcal{A}\phi)(x) = \int_a^b K(x,t)\phi(t)dt, \tag{1}$$

where the kernel $K(\cdot,\cdot)$ is continuous and hence square-integrable function on $[a;b]\times[a;b]$, is Fredholm. The operator of the form (1) is also called a linear Fredholm integral operator with the kernel $K(\cdot,\cdot)$. In the present note we considered the case where the kernel $K(\cdot,\cdot)$ is degenerate.

Let $\mathbb{T}^{\mathrm{d}}$ be the d-dimensional torus and $L_2(\mathbb{T}^{\mathrm{d}})$ be the Hilbert space of square integrable symmetric (complex) functions defined on $\mathbb{T}^{\mathrm{d}}$.

In the Hilbert space $L_2(\mathbb{T}^{\mathrm{d}})$ we consider the Fredholm integral operators of the form

$$(A_{ij}f_j)(x) = a_{ji}(x)\int_{\mathbb{T}^{\mathrm{d}}} a_{ij}(t)f_j(t)dt, \quad f_i \in L_2(\mathbb{T}^{\mathrm{d}}), \quad i \leq j, \quad i,j=1,2,3,$$

where $a_{ij}(\cdot)$, $i,j=1,2,3$ are the real-valued continuous functions on $\mathbb{T}^{\mathrm{d}}$. Then it is easy to see that $A_{ij}^* = A_{ji}$ for all $i,j=1,2,3$.

First we investigate the spectrum of $\mathcal{A}_1 := A_{11}$. Direct calculations show that the operator $\mathcal{A}_1$ has a purely point spectrum and the equality $\sigma_{\mathrm{pp}}(\mathcal{A}_1) = \{0, \|a_{11}\|^2\}$ holds, where the number $\lambda=0$ is an eigenvalue of $\mathcal{A}_1$ with infinite multiplicity, the number $\lambda = \|a_{11}\|^2$ is a simple eigenvalue of $\mathcal{A}_1$.

For the further discussions we denote

$$L_2^{(2)}(\mathbb{T}^{\mathrm{d}}) := \{f=(f_1,f_2) : \, f_\alpha \in L_2(\mathbb{T}^{\mathrm{d}}), \, \alpha=1,2\},$$
$$L_2^{(3)}(\mathbb{T}^{\mathrm{d}}) := \{f=(f_1,f_2,f_3) : \, f_\alpha \in L_2(\mathbb{T}^{\mathrm{d}}), \, \alpha=1,2,3\}.$$

Notice that the norm and scalar product in $L_2^{(3)}(\mathbb{T}^{\mathrm{d}})$ are defined as

$$\|f\| = \left(\int_{\mathbb{T}^{\mathrm{d}}} |f_1(t)|^2dt + \int_{\mathbb{T}^{\mathrm{d}}} |f_2(t)|^2dt + \int_{\mathbb{T}^{\mathrm{d}}} |f_3(t)|^2dt\right)^{1/2};$$
$$(f,g) = \int_{\mathbb{T}^{\mathrm{d}}} f_1(t)\overline{g_1(t)}dt + \int_{\mathbb{T}^{\mathrm{d}}} f_2(t)\overline{g_2(t)}dt + \int_{\mathbb{T}^{\mathrm{d}}} f_3(t)\overline{g_3(t)}dt$$

for $f=(f_1,f_2,f_3), g=(g_1,g_2,g_3) \in L_2^{(3)}(\mathbb{T}^{\mathrm{d}})$.

For $n=2,3$ in the Hilbert space $L_2^{(n)}(\mathbb{T}^{\mathrm{d}})$ we consider the following $n\times n$ operator matrix

$$\mathcal{A}_2 := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathcal{A}_3 := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Under these assumptions the operator matrix $\mathcal{A}_\alpha$ is bounded and self-adjoint in $L_2^{(\alpha)}(\mathbb{T}^{\mathrm{d}})$ for $\alpha=2,3$.

Operators of this type are arise in the process of constructing the Faddeev equations for the eigenfunctions of the model operators corresponding to the Hamiltonians of a three-particle system on a lattice [1,2].

Note that all matrix elements A_{ij} of $\mathcal{A}_3$ are one-dimensional operators, and hence depending on the functions $a_{ij}(\cdot)$, $i,j=1,2,3$ the operator matrix $\mathcal{A}_3$ is an at most 9-dimensional operator. Analogously, the operator matrix $\mathcal{A}_2$ is an at most 4-dimensional operator. Since $L_2(\mathbb{T}^{\rm d})$, $L_2^{(2)}(\mathbb{T}^{\rm d})$ and $L_2^{(3)}(\mathbb{T}^{\rm d})$ are the infinite-dimensional Hilbert spaces, that is,

$$\dim L_2(\mathbb{T}^{\rm d})=\dim L_2^{(2)}(\mathbb{T}^{\rm d})=\dim L_2^{(3)}(\mathbb{T}^{\rm d})=\infty,$$

the equalities hold:

$$\sigma_{\rm ess}(\mathcal{A}_1)=\sigma_{\rm ess}(\mathcal{A}_2)=\sigma_{\rm ess}(\mathcal{A}_3)=\{0\}.$$

To study the non zero eigenvalues of the operator matrices $\mathcal{A}_\alpha$, $\alpha=2,3$ we introduce the following functions:

$$\Delta_2(\lambda):=\begin{vmatrix}\Delta_{11}(\lambda) & \Delta_{12} & 0 & 0\\ 0 & \Delta_{22}(\lambda) & \Delta_{14} & \Delta_{15}\\ \Delta_{12} & \Delta_{42} & \Delta_{33}(\lambda) & 0\\ 0 & 0 & \Delta_{25} & \Delta_{44}(\lambda)\end{vmatrix},$$

$$\Delta(\lambda):=\begin{vmatrix}\Delta_{11}(\lambda) & \Delta_{12} & \cdots & \Delta_{19}\\ \Delta_{21} & \Delta_{22}(\lambda) & \cdots & \Delta_{29}\\ \vdots & \vdots & \ddots & \vdots\\ \Delta_{91} & \Delta_{92} & \cdots & \Delta_{99}(\lambda)\end{vmatrix},$$

where the matrix elements are defined by

$$\begin{aligned}
&\Delta_{11}(\lambda):=\|a_{11}\|^2-\lambda,\quad \Delta_{12}:=(a_{11},a_{21}),\quad \Delta_{13}:=(a_{11},a_{31});\\
&\Delta_{22}(\lambda):=-\lambda,\quad \Delta_{24}:=\|a_{12}\|^2,\quad \Delta_{25}:=(a_{12},a_{22}),\quad \Delta_{26}:=(a_{12},a_{32});\\
&\Delta_{33}(\lambda):=-\lambda,\quad \Delta_{37}:=\|a_{13}\|^2,\quad \Delta_{38}:=(a_{13},a_{23}),\quad \Delta_{39}:=(a_{13},a_{33});\\
&\Delta_{41}:=(a_{21},a_{11}),\quad \Delta_{42}:=\|a_{21}\|^2,\quad \Delta_{43}:=(a_{21},a_{31}),\quad \Delta_{44}(\lambda):=-\lambda;\\
&\Delta_{54}:=(a_{22},a_{12}),\quad \Delta_{55}:=\|a_{22}\|^2-\lambda,\quad \Delta_{56}:=(a_{22},a_{32});\\
&\Delta_{66}(\lambda):=-\lambda,\quad \Delta_{67}:=(a_{23},a_{13}),\quad \Delta_{68}:=\|a_{23}\|^2,\quad \Delta_{69}:=(a_{23},a_{33});\\
&\Delta_{71}:=(a_{31},a_{11}),\quad \Delta_{72}:=(a_{31},a_{21}),\quad \Delta_{73}:=\|a_{31}\|^2,\quad \Delta_{77}(\lambda):=-\lambda;\\
&\Delta_{84}:=(a_{32},a_{12}),\quad \Delta_{85}:=(a_{32},a_{22}),\quad \Delta_{86}:=\|a_{32}\|^2,\quad \Delta_{88}(\lambda):=-\lambda;\\
&\Delta_{97}:=(a_{33},a_{13}),\quad \Delta_{98}:=(a_{33},a_{23}),\quad \Delta_{86}:=\|a_{32}\|^2,\quad \Delta_{99}(\lambda):=\|a_{33}\|^2-\lambda\\
&\Delta_{ij}=0,\quad \text{otherwise.}
\end{aligned}$$

In the following theorem we describe the point spectrum of $\mathcal{A}_\alpha$, $\alpha=2,3$.

Theorem 1. *For $\alpha=2,3$ the operator matrix $\mathcal{A}_\alpha$ has a purely point spectrum and*

$$\sigma_{\rm pp}(\mathcal{A}_\alpha)=\{0\}\cup\{\lambda\in\mathbb{R}:\Delta_\alpha(\lambda)=0\}.$$

Moreover, the number $\lambda=0$ is an eigenvalue of $\mathcal{A}_\alpha$ with infinite multiplicity.

It can be seen that the function $\Delta_2(\cdot)$ is a polynomial of order 4 with respect to λ. Therefore, it has at most 4 real zeros (taking into account the multiplicity). Therefore, by virtue of Theorem 1, an operator matrix $\mathcal{A}_2$ can have at most 4 (taking into account the multiplicity) eigenvalues with finite multiplicity. Analogously, an operator matrix $\mathcal{A}_3$ can have at most 9 (taking into account the multiplicity) eigenvalues with finite multiplicity.

Using Theorem 1 and the fact about $\sigma_{\rm pp}(\mathcal{A}_1)$ it is possible to find an exact representation of the numerical range of the operator $\mathcal{A}_\alpha$, $\alpha=1,2,3$. It should be noted that since the operator $\mathcal{A}_\alpha$, $\alpha=1,2,3$ has a purely point spectrum, its numerical range $W(\mathcal{A}_\alpha)$ always a bounded (closed) segment and for $\alpha=1,2,3$ the equality

$$W(\mathcal{A}_\alpha)=[\min\sigma_{\rm pp}(\mathcal{A}_\alpha);\max\sigma_{\rm pp}(\mathcal{A}_\alpha)]$$

is valid. In particular, we have $W(\mathcal{A}_1)=\left[0;\|a_{11}\|^2\right]$. The study of quadratic numerical range of $\mathcal{A}_2$ and cubic numerical range of $\mathcal{A}_3$ needs an additional investigations.

Reference

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