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POINT SPECTRUM OF THE OPERATOR MATRICES WITH THE FREDHOLM INTEGRAL OPERATORS

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Abstract. *In the present paper we consider* 2×2 *and* 3×3 *operator matrices, where the matrix elements are Fredholm integral operators. It is shown that these matrices has only the purely point spectrum.*

Key words: *block operator matrix, Fredholm integral operator, point spectrum, linear operator, eigenvalue.*

ТОЧЕЧНЫЙ СПЕКТР ОПЕРАТОРНЫХ МАТРИЦ С ИНТЕГРАЛЬНЫМИ ОПЕРАТОРАМИ ФРЕДГОЛЬМА

Аннотация. *В настоящей работе рассматриваются и операторные матрицы, матричными элементами которых являются интегральные операторы Фредгольма. Показано, что эти матрицы имеют только чисто точечный спектр.*

Ключевые слова: *блочно-операторная матрица, интегральный оператор Фредгольма, точечный спектр, линейный оператор, собственное значение.*

FREDHOLM INTEGRAL OPERATORLARI BILAN OPERATOR MATRITASINING NOKTA SPEKTRIMI

Annotatsiya. *Ushbu maqolada biz operator matritsalarini ko'rib chiqdilk. Bu matritsa elementlari Fredgolm integral operatorlari hisoblanadi. Ushbu matritsalar faqat nuqta spektriga ega ekanligi ko'rsatilgan.*

Kalit so'zlar: *blok operator matritsasi, Fredgolm integral operatori, nuqta spektri, chiziqli operator, xos qiymat.*

Introduction. Block operator matrices are matrices the entries which are linear operators between Banach or Hilbert spaces [1]. They arise in various areas of mathematics and its applications. One of the important class of the block operator matrices are the energy operators of a system of n non conserved number of particles. Such systems occur widely in mathematical physics, e.g. statistical physics $[2-8]$, solid-state physics [6] and the theory of quantum fields [6,7].

In the theory of solid-state physics [6], quantum field theory [7] and statistical physics $[2-8]$ some important problems arise where the number of quasi-particles is not fixed. The study of systems with a non conserved, but bounded, number of particles is reduced to the study of the spectral properties of self-adjoint operators acting in "the cut" subspace $\mathcal{H}^{(n)}$, consisting of on particle, two particle and n- particle subspaces of the Fock space [2,7,8]

In mathematics, Fredholm operators are certain operators that arise in the Fredholm theory of integral equations. They are named in honour of Erik Ivar Fredholm.

A linear operator $\mathcal{A}$ from a Banach space X to a Banach space Y is called a Fredholm operator if

1. $\mathcal{A}$ is closed;
2. the domain $D(\mathcal{A})$ of $\mathcal{A}$ is dense in X;
3. $\alpha(\mathcal{A})$, the dimension of the null space $N(\mathcal{A})$ of $\mathcal{A}$, is finite;
4. $R(\mathcal{A})$, the range of $\mathcal{A}$, is closed in Y;
5. $\beta(\mathcal{A})$, the codimension of $R(\mathcal{A})$ in Y, is finite.

In particular, on the spaces $C[a;b]$ or $L_2[a;b]$ an operator of the form

$$(\mathcal{A}\phi)(x)=\int_a^b K(x,t)\phi(t)dt, \qquad (1)$$

where the kernel $K(\cdot,\cdot)$ is continuous and hence square-integrable function on $[a;b]\times[a;b]$, is Fredholm. The operator of the form (1) is also called a linear Fredholm integral operator with the kernel $K(\cdot,\cdot)$. In the present paper we considered the case where the kernel $K(\cdot,\cdot)$ is degenerate.

In the present paper for $n=2,3$ we consider $n\times n$ block operator matrix $\mathcal{A}_n$ with the Fredholm integral operator.

We obtain the following results:

-determined the point spectrum of the operator matrices $\mathcal{A}_\alpha, \alpha=1,2,3$ with the Fredholm integral operatoprs;

-$\mathcal{A}_\alpha, \alpha=1,2,3$ important spectrum for operator matrices are studied;

- Determinants corresponding to matrices with operator $\mathcal{A}_\alpha, \alpha=1,2,3$ are constructed
- The numerical range of matrices with operator $\mathcal{A}_\alpha, \alpha=1,2,3$ is defined

$n\times n(n=2,3)$ operator matrices with the Fredholm integral operators

Let $\mathbb{T}^d$ be the d -dimensional torus and $L_2(\mathbb{T}^d)$ be the Hilbert space of square integrable symmetric (complex) functions defined on $\mathbb{T}^d$.

In the Hilbert space $L_2(\mathbb{T}^d)$ we consider the Fredholm integral operators of the form

$$(A_{ij}f_j)(x)=a_{ji}(x)\int_{\mathbb{T}^d}a_{ij}(t)f_j(t)dt, \qquad f_i\in L_2(\mathbb{T}^d), \qquad i\le j, \qquad i,j=1,2,3,$$

where $a_{ij}(\cdot), i,j=1,2,3$ are the real-valued continuous functions on $\mathbb{T}^d$. Then it is easy to see that $A_{ij}^*=A_{ji}$ for all $i,j=1,2,3$.

First we investigate the spectrum of $\mathcal{A}_1:=A_{11}$. Direct calculations show that the operator $\mathcal{A}_1$ has a purely point spectrum and the equality $\sigma_{pp}(\mathcal{A}_1)=\{0,\|a_{11}\|^2\}$ holds, where the number $\lambda=0$ is an eigenvalue of $\mathcal{A}_1$ with infinite multiplicity, the number $\lambda=\|a_{11}\|^2$ is a simple eigenvalue of $\mathcal{A}_1$.

For the further discussions we denote

$$L_2^{(2)}(\mathbb{T}^d):=\{f=(f_1,f_2): f_\alpha\in L_2(\mathbb{T}^d), \alpha=1,2\}$$
$$L_2^{(3)}(\mathbb{T}^d):=\{f=(f_1,f_2,f_3): f_\alpha\in L_2(\mathbb{T}^d), \alpha=1,2,3\}.$$

Notice that the norm and scalar product in $L_2^{(3)}(\mathbb{T}^d)$ are defined as

$$\|f\|=\left(\int_{\mathbb{T}^d}|f_1(t)|^2dt+\int_{\mathbb{T}^d}|f_2(t)|^2dt+\int_{\mathbb{T}^d}|f_3(t)|^2dt\right)^{1/2};$$

$$(f,g)=\int_{\mathbb{T}^d}f_1(t)\overline{g_1(t)}dt+\int_{\mathbb{T}^d}f_2(t)\overline{g_2(t)}dt+\int_{\mathbb{T}^d}f_3(t)\overline{g_3(t)}dt$$

for $f=(f_1,f_2,f_3),\ g=(g_1,g_2,g_3)\in L_2^{(3)}(\mathbb{T}^d)$

For $n=2,3$ in the Hilbert space $L_2^{(n)}(\mathbb{T}^d)$ we consider the following $n\times n$ operator matrix

$$\mathcal{A}_2:=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix},\quad \mathcal{A}_3:=\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix}.$$

Under these assumptions the operator matrix $\mathcal{A}_\alpha$ is bounded and self-adjoint in $L_2^{(\alpha)}(\mathbb{T}^d)$ for $\alpha=2,3$.

Operators of this type are arise in the process of constructing the Faddeev equations for the eigenfunctions of the model operators corresponding to the Hamiltonians of a three-particle system on a lattice [9,10].

Note that all matrix elements A_{ij} of $\mathcal{A}_3$ are one-dimensional operators, and hence depending on the functions $a_{ij}(\cdot), i,j=1,2,3$ the operator matrix $\mathcal{A}_3$ is an at most 9-dimensional operator. Analogously, the operator matrix $\mathcal{A}_2$ is an at most 4-dimensional operator. Since $L_2(\mathbb{T}^d)$, $L_2^{(2)}(\mathbb{T}^d)$ and $L_2^{(3)}(\mathbb{T}^d)$ are the infinite-dimensional Hilbert spaces, that is,

$dimL_2(\mathbb{T}^d)=dimL_2^{(2)}(\mathbb{T}^d)=dimL_2^{(3)}(\mathbb{T}^d)=\infty$,

the equalities hold:

$\sigma_{ess}(\mathcal{A}_1) = \sigma_{ess}(\mathcal{A}_2) = \sigma_{ess}(\mathcal{A}_3) = \{0\}$

To study the non zero eigenvalues of the operator matrices $\mathcal{A}_\alpha$, $\alpha = 2,3$ we introduce the following functions:

$$\Delta_2(\lambda) := \begin{vmatrix} \Delta_{11}(\lambda) & \Delta_{12} & 0 & 0 \\ 0 & \Delta_{22}(\lambda) & \Delta_{24} & \Delta_{25} \\ \Delta_{12} & \Delta_{42} & \Delta_{33}(\lambda) & 0 \\ 0 & 0 & \Delta_{25} & \Delta_{55}(\lambda) \end{vmatrix},$$

$$\Delta_3(\lambda) := \begin{vmatrix} \Delta_{11}(\lambda) & \Delta_{12} & \cdots & \Delta_{19} \\ \Delta_{21} & \Delta_{22}(\lambda) & \cdots & \Delta_{29} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{91} & \Delta_{92} & \cdots & \Delta_{99}(\lambda) \end{vmatrix},$$

where the matrix elements are defined by

$$\Delta_{11}(\lambda) := \| a_{11} \|^2 - \lambda, \quad \Delta_{12} := (a_{11}, a_{21}), \quad \Delta_{13} := (a_{11}, a_{31});$$
$$\Delta_{22}(\lambda) := -\lambda, \quad \Delta_{24} := \| a_{12} \|^2, \quad \Delta_{25} := (a_{12}, a_{22}), \quad \Delta_{26} := (a_{12}, a_{32});$$
$$\Delta_{33}(\lambda) := -\lambda, \quad \Delta_{37} := \| a_{13} \|^2, \quad \Delta_{38} := (a_{13}, a_{23}), \quad \Delta_{39} := (a_{13}, a_{33});$$
$$\Delta_{41} := (a_{21}, a_{11}), \quad \Delta_{42} := \| a_{21} \|^2, \quad \Delta_{43} := (a_{21}, a_{31}), \quad \Delta_{44}(\lambda) := -\lambda;$$
$$\Delta_{54} := (a_{22}, a_{12}), \quad \Delta_{55} := \| a_{22} \|^2 - \lambda, \quad \Delta_{56} := (a_{22}, a_{32});$$
$$\Delta_{66}(\lambda) := -\lambda, \quad \Delta_{67} := (a_{23}, a_{13}), \quad \Delta_{68} := \| a_{23} \|^2, \quad \Delta_{69} := (a_{23}, a_{33});$$
$$\Delta_{71} := (a_{31}, a_{11}), \quad \Delta_{72} := (a_{31}, a_{21}), \quad \Delta_{73} := \| a_{31} \|^2, \quad \Delta_{77}(\lambda) := -\lambda;$$
$$\Delta_{84} := (a_{32}, a_{12}), \quad \Delta_{85} := (a_{32}, a_{22}), \quad \Delta_{86} := \| a_{32} \|^2, \quad \Delta_{88}(\lambda) := -\lambda;$$
$$\Delta_{97} := (a_{33}, a_{13}), \quad \Delta_{98} := (a_{33}, a_{23}), \quad \Delta_{86} := \| a_{32} \|^2, \quad \Delta_{99}(\lambda) := \| a_{33} \|^2 - \lambda$$
$$\Delta_{ij} = 0, \quad \text{otherwise}.$$

First we discuss the case $\alpha = 2$. In order to study the non-zero eigenvalues of $\mathcal{A}_2$ we consider the eigenvalue equation $\mathcal{A}_2 f = \lambda f, f = (f_1, f_2)$ or

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \end{pmatrix}.$$

We write this equation in the form of the following system of equations:

$$\begin{cases} A_{11} f_1 + A_{12} f_2 = \lambda f_1; \\ A_{21} f_1 + A_{22} f_2 = \lambda f_2. \end{cases} \qquad (2)$$

Using the definition of operators A_{ij}, we transform system (2) into an equivalent system:

$$\begin{cases} a_{11}(x) \int_{\mathbb{T}^d} a_{11}(t) f_1(t) dt + a_{21}(x) \int_{\mathbb{T}^d} a_{12}(t) f_2(t) dt = \lambda f_1(x); \\ a_{12}(x) \int_{\mathbb{T}^d} a_{21}(t) f_1(t) dt + a_{22}(x) \int_{\mathbb{T}^d} a_{22}(t) f_2(t) dt = \lambda f_2(x). \end{cases} \qquad (3)$$

We introduce the following definition:

$$c_{ij} = \int_{\mathbb{T}^d} a_{ij}(t) f_j(t) dt, \qquad i,j = 1,2 \qquad (4)$$

then system (3) becomes

$$\begin{cases} a_{11}(x) c_{11} + a_{21}(x) c_{12} = \lambda f_1(x); \\ a_{12}(x) c_{21} + a_{22}(x) c_{22} = \lambda f_2(x). \end{cases} \qquad (5)$$

(5) dividing both equations of the system by the number $\lambda \neq 0$, we find the following expressions for $f_1(x)$ and $f_2(x)$:

$$\begin{cases} f_1(x) = \frac{1}{\lambda} [a_{11}(x) c_{11} + a_{21}(x) c_{12}]; \\ f_2(x) = \frac{1}{\lambda} [a_{12}(x) c_{21} + a_{22}(x) c_{22}]. \end{cases} \qquad (6)$$

we put equalities (6) in (4):

$$c_{11} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{11}{}^2(t)c_{11} + a_{11}(t) \cdot a_{21}(t)c_{12}]dt;$$
$$c_{12} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{12}{}^2(t)c_{21} + a_{12}(t) \cdot a_{22}(t)c_{22}]dt;$$
$$c_{21} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{21}{}^2(t)c_{12} + a_{11}(t) \cdot a_{21}(t)c_{11}]dt;$$
$$c_{22} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{22}{}^2(t)c_{22} + a_{12}(t) \cdot a_{22}(t)c_{21}]dt.$$

The resulting system of equations is reduced to the following system of homogeneous equations:

$$\begin{cases} (\|a_{11}\|^2 - \lambda)c_{11} + (a_{11}, a_{21})c_{12} = 0 \\ -\lambda c_{12} + \|a_{12}\|^2 c_{21} + (a_{12}, a_{22})c_{22} = 0 \\ (a_{11}, a_{21})c_{11} + \|a_{21}\|^2 c_{12} - \lambda c_{21} = 0 \\ (a_{12}, a_{22})c_{21} + (\|a_{22}\|^2 - \lambda)c_{22} = 0 \end{cases} \qquad (7)$$

Similarly, $n = 3$ cases are analyzed. In order to study the non-zero eigenvalues of the operator matrix $\mathcal{A}_3$, we consider the eigenvalue equation $\mathcal{A}_3 f = \lambda f,\ f = (f_1, f_2, f_3)$, or:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \\ \lambda f_3 \end{pmatrix}.$$

From this, we write down the following system of equations:

$$\begin{cases} A_{11}f_1 + A_{12}f_2 + A_{13}f_3 = \lambda f_1 \\ A_{21}f_1 + A_{22}f_2 + A_{23}f_3 = \lambda f_2 \\ A_{31}f_1 + A_{32}f_2 + A_{33}f_3 = \lambda f_3 \end{cases}$$

From the definition of operators A_{ij} system

$$\begin{cases} a_{11}(x) \int_{\mathbb{T}^d} a_{11}(t)f_1(t)dt + a_{21}(x) \int_{\mathbb{T}^d} a_{12}(t)f_2(t)dt + a_{31}(x) \int_{\mathbb{T}^d} a_{13}(t)f_3(t)dt = \lambda f_1(x); \\ a_{12}(x) \int_{\mathbb{T}^d} a_{21}(t)f_1(t)dt + a_{22}(x) \int_{\mathbb{T}^d} a_{22}(t)f_2(t)dt + a_{32}(x) \int_{\mathbb{T}^d} a_{23}(t)f_3(t)dt = \lambda f_2(x) \\ a_{13}(x) \int_{\mathbb{T}^d} a_{31}(t)f_1(t)dt + a_{23}(x) \int_{\mathbb{T}^d} a_{32}(t)f_2(t)dt + a_{33}(x) \int_{\mathbb{T}^d} a_{33}(t)f_3(t)dt = \lambda f_3(x) \end{cases}$$

appears. By assigning $c_{ij}, (i,j = 1,2,3)$ to this system of equations, we write the following equivalent system:

$$\begin{cases} a_{11}(x)c_{11} + a_{21}(x)c_{12} + a_{31}(x)c_{13} = \lambda f_1(x); \\ a_{12}(x)c_{21} + a_{22}(x)c_{22} + a_{32}(x)c_{23} = \lambda f_2(x); \\ a_{13}(x)c_{31} + a_{23}(x)c_{32} + a_{33}(x)c_{33} = \lambda f_3(x). \end{cases}$$

We divide by the number $\lambda \neq 0$:

$$\begin{cases} f_1(x) = \frac{1}{\lambda}[a_{11}(x)c_{11} + a_{21}(x)c_{12} + a_{31}(x)c_{13}]; \\ f_2(x) = \frac{1}{\lambda}[a_{12}(x)c_{21} + a_{22}(x)c_{22} + a_{32}(x)c_{23}]; \\ f_3(x) = \frac{1}{\lambda}[a_{13}(x)c_{31} + a_{23}(x)c_{32} + a_{33}(x)c_{33}]. \end{cases} \qquad (8)$$

Let's set $f_1(x), f_2(x), f_3(x)$ found in expression (8) to c_{ij}:

$$c_{11} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{11}}^2(t)c_{11} + a_{11}(t) \cdot a_{21}(t)c_{12} + a_{11}(t) \cdot a_{31}(t)c_{13}]dt;$$
$$c_{12} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{12}}^2(t)c_{21} + a_{12}(t) \cdot a_{22}(t)c_{22} + a_{12}(t) \cdot a_{32}(t)c_{23}]dt;$$
$$c_{13} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{13}}^2(t)c_{31} + a_{13}(t) \cdot a_{23}(t)c_{32} + a_{13}(t) \cdot a_{33}(t)c_{33}]dt;$$
$$c_{21} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{21}}^2(t)c_{12} + a_{11}(t) \cdot a_{21}(t)c_{11} + a_{21}(t) \cdot a_{31}(t)c_{13}]dt;$$
$$c_{22} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{22}}^2(t)c_{22} + a_{12}(t) \cdot a_{22}(t)c_{21} + a_{22}(t) \cdot a_{32}(t)c_{23}]dt;$$
$$c_{23} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{23}}^2(t)c_{32} + a_{23}(t) \cdot a_{13}(t)c_{31} + a_{23}(t) \cdot a_{33}(t)c_{33}]dt;$$
$$c_{31} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{31}}^2(t)c_{13} + a_{31}(t) \cdot a_{11}(t)c_{11} + a_{31}(t) \cdot a_{21}(t)c_{12}]dt;$$
$$c_{32} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{32}}^2(t)c_{23} + a_{12}(t) \cdot a_{32}(t)c_{21} + a_{22}(t) \cdot a_{32}(t)c_{22}]dt;$$
$$c_{33} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [{a_{33}}^2(t)c_{33} + a_{13}(t) \cdot a_{33}(t)c_{31} + a_{23}(t) \cdot a_{33}(t)c_{32}]dt.$$

The following system of homogeneous equations is written from the resulting system of equations:

$$\begin{cases} (\|a_{11}\|^2 - \lambda)c_{11} + (a_{11}, a_{21})c_{12} + (a_{11}, a_{31})c_{13} = 0; \\ -\lambda c_{12} + (a_{12}, a_{22})c_{22} + (a_{12}, a_{32})c_{23} = 0; \\ -\lambda c_{13} + \|a_{13}\|^2 c_{31} + (a_{13}, a_{23})c_{32} + (a_{13}, a_{32})c_{33} = 0; \\ (a_{11}, a_{21})c_{11} + \|a_{21}\|^2 c_{12} + (a_{21}, a_{31})c_{13} - \lambda c_{21} = 0; \\ (a_{12}, a_{22})c_{21} + (\|a_{22}\|^2 - \lambda)c_{22} + (a_{22}, a_{32})c_{23} = 0; \\ -\lambda c_{23} + (a_{23}, a_{13})c_{31} + \|a_{23}\|^2 c_{32} + (a_{23}, a_{33})c_{33} = 0; \\ (a_{11}, a_{31})c_{11} + (a_{31}, a_{21})c_{12} + \|a_{31}\|^2 c_{13} - \lambda c_{31} = 0; \\ (a_{12}, a_{32})c_{21} + (a_{32}, a_{22})c_{22} + \|a_{32}\|^2 c_{23} - \lambda c_{32} = 0; \\ (a_{13}, a_{33})c_{31} + (a_{33}, a_{23})c_{32} + (\|a_{33}\|^2 - \lambda)c_{33} = 0. \end{cases}$$

In the following theorem we describe the point spectrum of $\mathcal{A}_\alpha$, $\alpha = 2,3$.

Theorem 1. *For $\alpha = 2,3$ the operator matrix $\mathcal{A}_\alpha$ has a purely point spectrum and*
$\sigma_{pp}(\mathcal{A}_\alpha) = \{0\} \cup \{\lambda \in \mathbb{R}: \Delta_\alpha(\lambda) = 0\}$.
Moreover, the number $\lambda = 0$ is an eigenvalue of $\mathcal{A}_\alpha$ with infinite multiplicity.

It can be seen that the function $\Delta_2(\cdot)$ is a polynomial of order 4 with respect to λ. Therefore, it has at most 4 real zeros (taking into account the multiplicity). Therefore, by virtue of Theorem 1, an operator matrix $\mathcal{A}_2$ can have at most 4 (taking into account the multiplicity) eigenvalues with finite multiplicity. Analogously, an operator matrix $\mathcal{A}_3$ can have at most 9 (taking into account the multiplicity) eigenvalues with finite multiplicity.

Using Theorem 1 and the fact about $\sigma_{pp}(\mathcal{A}_1)$ it is possible to find an exact representation of the numerical range of the operator $\mathcal{A}_\alpha$, $\alpha = 1,2,3$. It should be noted that since the operator $\mathcal{A}_\alpha$, $\alpha = 1,2,3$ has a purely point spectrum, its numerical range $W(\mathcal{A}_\alpha)$ always a bounded (closed) segment and for $\alpha = 1,2,3$ the equality

$$W(\mathcal{A}_\alpha) = [\min \sigma_{pp}(\mathcal{A}_\alpha); \max \sigma_{pp}(\mathcal{A}_\alpha)]$$

is valid. In particular, we have $W(\mathcal{A}_1) = [0; \|a_{11}\|^2]$. The study of quadratic numerical range of $\mathcal{A}_2$ and cubic numerical range of $\mathcal{A}_3$ needs an additional investigations.

$\mathcal{A}_1$ operator resolvent. In order to determine the resolvent operator of the operator $\mathcal{A}_1$, we consider the equation

$(\mathcal{A}_1 - \lambda)f = g$

with respect to the functions $f, g \in L_2(\mathbb{T}^d)$ for the number λ.

$\mathcal{A}_1$ is the last equality according to the definition of the operator

$a_{11}(x) \cdot c_{11} - \lambda f(x) = g(x)$

appears. Here, the value of c_{11} is determined using equation (4). From the above equation, we find the following expression for the function $f(x)$:

$$f(x) = \frac{1}{\lambda}[a_{11}(x) \cdot c_{11} - g(x)] \qquad (9)$$

We put the expression (9) found for $f(x)$ in (4), and

$$c_{11} = \frac{1}{\lambda}\int_{\mathbb{T}^d} a_{11}(t)[a_{11}(t) \cdot c_{11} - g(t)]dt$$

we will have equality. From this we find c_{11} and put it in the expression found for $f(x)$ and

$$f(x) = \frac{1}{\lambda}\left(\frac{a_{11}(x)}{\| a_{11} \|^2 - \lambda}(a_{11}, g) - g(x)\right)$$

we form the equation. The expression on the right side of this equality indicates the action formula of the resolvent operator $R_{\mathcal{A}_1}(\lambda)$ corresponding to the operator $\mathcal{A}_1$. Which

$$R_{\mathcal{A}_1}(\lambda)g = \frac{1}{\lambda}\left(\frac{a_{11}(x)}{\| a_{11} \|^2 - \lambda}\int_{\mathbb{T}^d} a_{11}(t) \cdot g_1(t)dt - g_1(x)\right)$$

$\mathcal{A}_2$ operator resolvent. We define the resolvent operator of the matrix with operator $\mathcal{A}_2$. For this, we consider the following equation with respect to $f = (f_1, f_2)$, $g = (g_1, g_2) \in L_2(\mathbb{T}^d)$ vector functions:

$(\mathcal{A}_2 - \lambda)f = g$

According to the definition of the matrix with operator $\mathcal{A}_2$, we write the following equation:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \lambda\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

We form the following system of equations from this equation:

$$\begin{cases} A_{11}f_1 + A_{12}f_2 - \lambda f_1 = g_1; \\ A_{21}f_1 + A_{22}f_2 - \lambda f_2 = g_2. \end{cases}$$

According to the definition of operators A_{ij}, this system of equations takes the following form:

$$\begin{cases} a_{11}(x)c_{11} + a_{21}(x)c_{12} - \lambda f_1(x) = g_1(x); \\ a_{12}(x)c_{21} + a_{22}(x)c_{22} - \lambda f_2(x) = g_2(x). \end{cases}$$

We can find $f_1(x)$ from the first equation of the last system of equations, and $f_2(x)$ from the second equation. Which

$$\begin{cases} f_1(x) = \frac{1}{\lambda}[a_{11}(x)c_{11} + a_{21}(x)c_{12} - g_1(x)]; \\ f_2(x) = \frac{1}{\lambda}[a_{12}(x)c_{21} + a_{22}(x)c_{22} - g_2(x)]. \end{cases}$$

Let's define the last expressions found in (4):

$$c_{11} = \frac{1}{\lambda}\cdot\int_{\mathbb{T}^d}[a_{11}{}^2(t)c_{11} + a_{11}(t)\cdot a_{21}(t)c_{12} - a_{11}(t)\cdot g_1(t)]dt;$$

$$c_{12} = \frac{1}{\lambda}\cdot\int_{\mathbb{T}^d}[a_{12}{}^2(t)c_{21} + a_{12}(t)\cdot a_{22}(t)c_{22} - a_{12}(t)\cdot g_2(t)]dt;$$

$$c_{21} = \frac{1}{\lambda}\cdot\int_{\mathbb{T}^d}[a_{21}{}^2(t)c_{12} + a_{11}(t)\cdot a_{21}(t)c_{11} - a_{21}(t)\cdot g_1(t)]dt;$$

$$c_{22} = \frac{1}{\lambda}\cdot\int_{\mathbb{T}^d}[a_{22}{}^2(t)c_{22} + a_{12}(t)\cdot a_{22}(t)c_{21} - a_{22}(t)\cdot g_2(t)]dt$$

or,

$$\begin{cases} (\| a_{11} \|^2-\lambda)c_{11}+(a_{11},a_{21})c_{12}-(a_{11},g_1)=0; \\ -\lambda c_{12}+\| a_{12} \|^2 c_{21}+(a_{12},a_{22})c_{22}-(a_{12},g_2)=0; \\ (a_{11},a_{21})c_{11}+\| a_{21} \|^2 c_{12}-\lambda c_{21}-(a_{21},g_1)=0; \\ (a_{12},a_{22})c_{21}+(\| a_{22} \|^2-\lambda)c_{22}-(a_{22},g_2)=0. \end{cases} \qquad (10)$$

we form a system of equations. We are looking for the solution of the system of equations (10). For this, we introduce the following definitions:

$$\Delta_2(\lambda)=\begin{vmatrix} \| a_{11} \|^2-\lambda & (a_{11},a_{21}) & 0 & 0 \\ 0 & -\lambda & \| a_{12} \|^2 & (a_{12},a_{22}) \\ (a_{11},a_{21}) & \| a_{21} \|^2 & -\lambda & 0 \\ 0 & 0 & (a_{12},a_{22}) & \| a_{22} \|^2-\lambda \end{vmatrix}; \quad G=\begin{pmatrix} (a_{11},g_1) \\ (a_{12},g_2) \\ (a_{21},g_1) \\ (a_{22},g_2) \end{pmatrix};$$

$$\Delta_2^{(1)}(\lambda)=\begin{vmatrix} (a_{11},g_1) & (a_{11},a_{21}) & 0 & 0 \\ (a_{12},g_2) & -\lambda & \| a_{12} \|^2 & (a_{12},a_{22}) \\ (a_{21},g_1) & \| a_{21} \|^2 & -\lambda & 0 \\ (a_{22},g_2) & 0 & (a_{12},a_{22}) & \| a_{22} \|^2-\lambda \end{vmatrix}=$$

$$=(a_{11},g_1)\cdot B_1(\lambda)-(a_{11},a_{21})\cdot G_1(\lambda);$$

$$\Delta_2^{(2)}(\lambda)=\begin{vmatrix} \| a_{11} \|^2-\lambda & (a_{11},g_1) & 0 & 0 \\ 0 & (a_{12},g_2) & \| a_{12} \|^2 & (a_{12},a_{22}) \\ (a_{11},a_{21}) & (a_{21},g_1) & -\lambda & 0 \\ 0 & (a_{22},g_2) & (a_{12},a_{22}) & \| a_{22} \|^2-\lambda \end{vmatrix}=$$

$$=(\| a_{11} \|^2-\lambda)\cdot G_1(\lambda)-(a_{11},g_1)\cdot B_2(\lambda);$$

$$\Delta_2^{(3)}(\lambda)=\begin{vmatrix} \| a_{11} \|^2-\lambda & (a_{11},a_{21}) & (a_{11},g_1) & 0 \\ 0 & -\lambda & (a_{12},g_2) & (a_{12},a_{22}) \\ (a_{11},a_{21}) & \| a_{21} \|^2 & (a_{21},g_1) & 0 \\ 0 & 0 & (a_{22},g_2) & \| a_{22} \|^2-\lambda \end{vmatrix}=$$

$$=(\| a_{22} \|^2-\lambda)\cdot G_2(\lambda)-(a_{22},g_2)\cdot B_3(\lambda);$$

$$\Delta_2^{(4)}(\lambda)=\begin{vmatrix} \| a_{11} \|^2-\lambda & (a_{11},a_{21}) & 0 & (a_{11},g_1) \\ 0 & -\lambda & \| a_{12} \|^2 & (a_{12},g_2) \\ (a_{21},a_{11}) & \| a_{21} \|^2 & -\lambda & (a_{21},g_1) \\ 0 & 0 & (a_{12},a_{22}) & (a_{22},g_2) \end{vmatrix}=$$

$$=(a_{22},g_2)\cdot B_4(\lambda)-(a_{12},a_{22})\cdot G_2(\lambda).$$

Here

$$B_1(\lambda)=\begin{vmatrix} -\lambda & \| a_{12} \|^2 & (a_{12},a_{22}) \\ \| a_{21} \|^2 & -\lambda & 0 \\ 0 & (a_{12},a_{22}) & \| a_{22} \|^2-\lambda \end{vmatrix}; \quad B_2(\lambda)=\begin{vmatrix} 0 & \| a_{12} \|^2 & (a_{12},a_{22}) \\ (a_{11},a_{21}) & -\lambda & 0 \\ 0 & (a_{12},a_{22}) & \| a_{22} \|^2-\lambda \end{vmatrix};$$

$$B_3(\lambda)=\begin{vmatrix} |a_{11} \|^2-\lambda & (a_{11},a_{21}) & 0 \\ 0 & -\lambda & (a_{12},a_{22}) \\ (a_{11},a_{21}) & \| a_{21} \|^2 & 0 \end{vmatrix}; \quad B_4(\lambda)=\begin{vmatrix} |a_{11} \|^2-\lambda & (a_{11},a_{21}) & 0 \\ 0 & -\lambda & \| a_{12} \|^2 \\ (a_{11},a_{21}) & \| a_{21} \|^2 & -\lambda \end{vmatrix};$$

$$G_1(\lambda)=\begin{vmatrix} (a_{12},g_2) & \| a_{12} \|^2 & (a_{12},a_{22}) \\ (a_{21},g_1) & -\lambda & 0 \\ (a_{22},g_2) & (a_{12},a_{22}) & |a_{22} \|^2-\lambda \end{vmatrix}; \quad G_2(\lambda)=\begin{vmatrix} |a_{11} \|^2-\lambda & (a_{11},a_{21}) & (a_{11},g_1) \\ 0 & -\lambda & (a_{12},g_2) \\ (a_{11},a_{21}) & \| a_{21} \|^2 & (a_{21},g_1) \end{vmatrix}.$$

The solutions of the system of equations (10) are as follows:

$$c_{11}=\frac{1}{\Delta_2(\lambda)}\cdot[(a_{11},g_1)\cdot B_1(\lambda)-(a_{11},a_{21})\cdot G_1(\lambda)];$$

$$c_{12}=\frac{1}{\Delta_2(\lambda)}\cdot[(\| a_{11}\|^2-\lambda)\cdot G_1(\lambda)-(a_{11},g_1)\cdot B_2(\lambda)];$$

$$c_{21}=\frac{1}{\Delta_2(\lambda)}\cdot[(\| a_{22}\|^2-\lambda)\cdot G_2(\lambda)-(a_{22},g_2)\cdot B_3(\lambda)];$$

$$c_{22}=\frac{1}{\Delta_2(\lambda)}\cdot[(a_{22},g_2)\cdot B_4(\lambda)-(a_{12},a_{22})\cdot G_2(\lambda)].$$

We put the expressions found for c_{ij} into the expressions found for $f_1(x)$ and $f_2(x)$ and form the following equations:

$$f_1(x)=\frac{1}{\lambda\cdot\Delta_2(\lambda)}\cdot[[a_{11}(x)\cdot B_1(\lambda)-a_{21}(x)\cdot B_2(\lambda)]\cdot(a_{11},g_1)+[a_{21}(x)\times$$
$$\times(\| a_{11}\|^2-\lambda)-a_{11}(x)\cdot(a_{11},a_{21})]\cdot[(a_{12},a_{22})^2-\| a_{12}\|^2\cdot(\| a_{22}\|^2-\lambda)]\cdot(a_{21},g_1)]+$$
$$+\frac{1}{\Delta_2(\lambda)}\cdot[a_{21}(x)\cdot(\| a_{11}\|^2-\lambda)-a_{11}(x)\cdot(a_{11},a_{21})]\times$$
$$\times[(a_{12},a_{22})\cdot(a_{22},g_2)-(\| a_{22}\|^2-\lambda)\cdot(a_{12},g_2)]-\frac{1}{\lambda}g_1(x);$$
$$f_2(x)=\frac{1}{\lambda\cdot\Delta_2(\lambda)}\cdot[[a_{22}(x)\cdot B_4(\lambda)-a_{12}(x)\cdot B_3(\lambda)]\cdot(a_{22},g_2)++[a_{12}(x)\times$$
$$\times(\| a_{22}\|^2-\lambda)-a_{22}(x)\cdot(a_{12},a_{22})]\cdot[(a_{11},a_{21})^2-\| a_{21}\|^2\cdot(\| a_{11}\|^2-\lambda)]\cdot(a_{12},g_2)]+$$
$$+\frac{1}{\Delta_2(\lambda)}\cdot[a_{12}(x)\cdot(\| a_{22}\|^2-\lambda)-a_{22}(x)\cdot(a_{12},a_{22})]\times$$
$$\times[(a_{11},a_{21})\cdot(a_{11},g_1)-(\| a_{11}\|^2-\lambda)\cdot(a_{21},g_1)]-\frac{1}{\lambda}g_2(x)$$

The expressions on the right side of the resulting equations indicate the action formula of the resolvent operator $R_{\mathcal{A}_2}(\lambda)$ corresponding to the operator matrix $\mathcal{A}_2$. Which,

$$R_{\mathcal{A}_2}(\lambda)=\begin{pmatrix}R_{11}(\lambda) & R_{12}(\lambda)\\ R_{21}(\lambda) & R_{22}(\lambda)\end{pmatrix}$$

$$R_{11}(\lambda)g_1=\frac{1}{\lambda\cdot\Delta_2(\lambda)}\cdot\Big[\big(a_{11}(x)\cdot B_1(\lambda)-a_{21}(x)\cdot B_2(\lambda)\big)\cdot\int_{\mathrm{T}^d}a_{11}(t)\cdot g_1(t)dt+$$
$$+\big(a_{21}(x)\cdot(\| a_{11}\|^2-\lambda)-a_{11}(x)\cdot(a_{11},a_{21})\big)\times$$
$$\times\left((a_{12},a_{22})^2-\| a_{12}\|^2\cdot(\| a_{22}\|^2-\lambda)\cdot\int_{\mathrm{T}^d}a_{21}(t)\cdot g_1(t)dt\right)\Big]-\frac{1}{\lambda}g_1(x);$$

$$R_{12}(\lambda)g_2=\frac{1}{\Delta_2(\lambda)}\cdot[a_{21}(x)\cdot(\| a_{11}\|^2-\lambda)-a_{11}(x)\cdot(a_{11},a_{21})]\times$$
$$\times\left[(a_{12},a_{22})\cdot\int_{\mathrm{T}^d}a_{22}(t)\cdot g_2(t)dt-(\| a_{22}\|^2-\lambda)\cdot\int_{\mathrm{T}^d}a_{12}(t)\cdot g_2(t)dt\right];$$
$$R_{21}(\lambda)g_1=\frac{1}{\Delta_2(\lambda)}\cdot[a_{12}(x)\cdot(\| a_{22}\|^2-\lambda)-a_{22}(x)\cdot(a_{12},a_{22})]\times$$
$$\times\left[(a_{11},a_{21})\cdot\int_{\mathrm{T}^d}a_{11}(t)\cdot g_1(t)dt-(\| a_{11}\|^2-\lambda)\right]\cdot\int_{\mathrm{T}^d}a_{21}(t)\cdot g_1(t)dt];$$

$$R_{22}(\lambda)g_2=\frac{1}{\lambda\cdot\Delta_2(\lambda)}\cdot\Big[\big(a_{22}(x)\cdot B_4(\lambda)-a_{12}(x)\cdot B_3(\lambda)\big)\cdot\int_{\mathrm{T}^d}a_{22}(t)\cdot g_2(t)dt+$$
$$+\big(a_{12}(x)\cdot(\| a_{22}\|^2-\lambda)-a_{22}(x)\cdot(a_{12},a_{22})\big)\times$$
$$\times\left((a_{11},a_{21})^2-\| a_{21}\|^2\cdot(\| a_{11}\|^2-\lambda)\cdot\int_{\mathrm{T}^d}a_{12}(t)\cdot g_2(t)dt\right)\Big]-\frac{1}{\lambda}g_2(x).$$

Conclusion. In this article, determined the point spectrum of the operator matrices $\mathcal{A}_\alpha,\alpha=1,2,3$ with the Fredholm integral operatoprs; $\mathcal{A}_\alpha,\alpha=1,2,3$ important spectrum for operator matrices are studied;

Determinants corresponding to matrices with operator $\mathcal{A}_\alpha, \alpha = 1,2,3$ are constructed; The numerical range of matrices with operator $\mathcal{A}_\alpha, \alpha = 1,2,3$ is determined and the resolvent operators of $\alpha = 1,2$ cases is found.

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