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ФАН ВА ТАЪЛИМ

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LOCAL INNER DERIVATIONS ON FOUR-DIMENSIONAL LIE ALGEBRAS

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Summary: In this paper we investigate local inner derivations of four-dimensional Lie algebras.

Key words: Lie algebra, derivation, inner derivation, local derivation, local inner derivation.

1. Introductions. Currently, various types of derivations over algebras are being studied, for example, derivations, local derivations, 2-local derivations, almost inner derivations, etc. Local derivations were first considered in the work of R.Kaidison in 1990 [11] and, independently in the work of D.Larson and A.Surur [12]. In these papers, some conditions are indicated under which local derivations is derivations. R.Kaidison's paper considered local derivations on von Neumann algebras and in some polynomial algebras. L.Molnár [13] introduced the definition of local inner derivations on standard operator algebras.

The first results concern to local and 2-local derivations and automorphisms on finite-dimensional Lie algebras over algebraically closed field of zero characteristic were obtained in [2, 3, 5] and [8]. Namely, in [5] it is proved that every 2-local derivation on a semi-simple Lie algebra L is a derivation and that each finite-dimensional nilpotent Lie algebra with dimension larger than two admits 2-local derivation which is not a derivation. In [2] the authors have proved that every local derivation on a semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. Concerning 2-local automorphism, Z.Chen and D.Wang in [8] prove that if L is a simple Lie algebra of type A_k , D_l or E_k , ($k = 6,7,8$) over an algebraically closed field of characteristic zero, then every 2-local automorphism of L is an automorphism. Finally, in [3] Sh.A.Ayupov and K.Kudaybergenov generalized this result of [8] and proved that every 2-local automorphism of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism. Moreover, they show also that every nilpotent Lie algebra with finite-dimension larger than two admits 2-local automorphisms which is not an automorphism. Local automorphisms of certain finite-dimensional simple Lie and Leibniz algebras are investigated in [4]. Almost inner derivations of Lie algebras were introduced by C.S. Gordon and E.N. Wilson [9] in the study of isospectral deformations of compact manifolds. Almost inner derivations of nilpotent, some solvable Lie algebras and some nilpotent Leibniz algebras were studied in the papers by [6] and [1].

2. Preliminaries. To begin with, recall the definition of Lie algebras.

Definition 2.1. An algebra L over field F is called a *Lie algebra* if its multiplication satisfies the identities:

- 1) $[x, x] = 0$,
- 2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$,

for all x, y, z in L .

The product $[x, y]$ is called the bracket of x and y . Identity 2) is called the Jacobi identity.

Let L be a finite-dimensional Lie algebra. For Lie algebra L we consider the following central and derived series:

$$L^1 = L, L^i = [L^{i-1}, L], i \geq 1,$$

$$L^{[1]} = L, L^{[k]} = [L^{[k-1]}, L^{[k-1]}], k \geq 1.$$

A Lie algebra L is *nilpotent (solvable)* if there exists $m \geq 1$ such that $L^m = 0$ ($L^{[m]} = 0$).

Definition 2.2. A *derivation* on a Lie algebra L is a linear map $D: L \rightarrow L$ which satisfies the Leibniz rule:

$$D([x, y]) = [D(x), y] + [x, D(y)] \tag{2.1}$$

for any $x, y \in L$. The set of all derivations of a Lie algebra denoted by $Der(L)$. Let a be an element of a Lie algebra L . Consider the operator of $ad_a : L \rightarrow L$ defined by $ad_a(x) = [x, a]$. This operator is a derivation and called *inner derivation*. The set of all inner derivations of a Lie algebra denoted by $InDer(L)$.

Definition 2.3. A linear operator Δ is called a *local derivation* if for any $x \in L$, there exists a derivation $D_x : L \rightarrow L$ (depending on x) such that $\Delta(x) = D_x(x)$.

Definition 2.4. A linear operator Δ is called a *local inner derivation* if for any $x \in L$, there exists a inner derivation $ad_x : L \rightarrow L$ (depending on x) such that $\Delta(x) = ad_x(x)$.

We present the following theorem which gives a classification of arbitrary four-dimensional Lie algebras.

Theorem 2.1. [7]. An arbitrary four-dimensional Lie algebra is isomorphic to one of the following algebras: L_0 : abelian;

$$L_1 : [e_1, e_2] = e_3;$$

$$L_2 : [e_1, e_2] = e_1;$$

$$L_3 : [e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3;$$

$$L_4 : [e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3, \lambda \in C^*, |\lambda| \leq 1;$$

$$L_5 : [e_1, e_2] = e_1, [e_3, e_4] = e_3;$$

$$L_6 : [e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2;$$

$$L_7 : [e_1, e_2] = e_3, [e_1, e_3] = e_4;$$

$$L_8 : [e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = \alpha e_4, \alpha \in C^*;$$

$$L_9 : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha e_2 - \beta e_3 - e_4, \alpha \in C^*, \beta \in C;$$

$$L_{10} : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha(e_2 + e_3), \alpha \in C^*;$$

$$L_{11} : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_2;$$

$$L_{12} : [e_1, e_2] = \frac{1}{3}e_2 + e_3, [e_1, e_3] = \frac{1}{3}e_3, [e_1, e_4] = \frac{1}{3}e_4;$$

$$L_{13} : [e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = 2e_4, [e_2, e_3] = e_4;$$

$$L_{14} : [e_1, e_2] = e_3, [e_1, e_3] = e_2, [e_2, e_3] = e_4;$$

$$L_{15} : [e_1, e_2] = e_3, [e_1, e_3] = -\alpha e_2 + e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, \alpha \in C;$$

3. Main results. In this section we will consider local inner derivations of four-dimensional Lie algebras.

The following theorem is the main result of this work.

Theorem 3.1. Any local inner derivation on the algebras $L_1 - L_{12}, L_{14}$ is an inner derivation, and on the algebras L_{13} and L_{15} there exists a local inner derivation which is not inner derivation.

Proof. We verify that local inner derivations on the algebras $L_1 - L_{12}, L_{14}$ are inner derivations.

The algebra L_0 . First, consider the algebra L_0 from Theorem 2.1. Inner derivations on the algebra L_0 are zero. Therefore, any local inner derivation is also zero.

The algebra L_1 . For the element $a = \sum_{i=1}^4 a_i e_i \in L_1$ we define the inner derivation on L_1 as follows

$$ad_a(x) = [x, a] = [x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4, a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4] = (x_1 a_2 - x_2 a_1) e_3$$

where $x = \sum_{i=1}^4 x_i e_i \in L_1$.

Let Δ be a local inner derivation of the algebra L_1 . By definition of local inner derivation, checking the equality $\Delta(e_i) = ad_{e_i}(e_i)$ ($i = 1, 2, 3, 4$) for the basis $\{e_1, e_2, e_3, e_4\}$, we obtain the following:

$$\Delta(e_1) = a_{21} e_3, \quad \Delta(e_2) = -a_{12} e_3, \quad \Delta(e_3) = \Delta(e_4) = 0.$$

Since the operator Δ is linear, then

$$\Delta(x) = \Delta(x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4) = (a_{21} x_1 - a_{12} x_2) e_3.$$

Then $\Delta(x) = [x, a_{12} e_1 + a_{21} e_2]$. This means that the operator Δ is an inner derivation.

The algebra L_2 . Repeating the previous technique for this algebra, we get that every local inner derivation on L_2 is a derivation.

For the algebras L_3 and L_4 , we check the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3$

The algebra L_3 . For the element $a = \sum_{i=1}^4 a_i e_i \in L_3$ we define the inner derivation on L_3 as follows

$$\begin{aligned} ad_a(x) &= [x, a] = [x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4, a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4] = \\ &= (x_1 a_2 + x_1 a_3 - x_2 a_1 - x_3 a_1) e_2 + (x_1 a_3 - x_3 a_1) e_3, \end{aligned}$$

where $x \in L_3$.

Let Δ be a local inner derivation of the algebra L_3 . Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3$, we obtain the following:

$$\begin{aligned} \Delta(e_1) &= (a_{21} + a_{31}) e_2 + a_{31} e_3, \\ \Delta(e_2) &= -a_{12} e_2, \\ \Delta(e_3) &= -a_{13} e_2 - a_{13} e_3, \\ \Delta(e_4) &= 0, \end{aligned} \tag{3.1}$$

$$\Delta(e_2 + e_3) = -2a_1 e_2 - a_1 e_3.$$

From the equality $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3)$ i.e. $-2a_1 e_2 - a_1 e_3 = -a_{12} e_2 - a_{13} e_2 - a_{13} e_3$ we get $a_{13} = a_{12}$. Substituting the resulting equality into (3.1) we have that

$$\Delta(e_3) = -a_{12} e_2 - a_{12} e_3. \tag{3.1'}$$

Then $\Delta(x) = [x, a_{12} e_1 + a_{21} e_2 + a_{31} e_3]$. This means that the operator Δ is an inner derivation.

The algebra L_4 . Repeating the previous technique for this algebra, we get that every local inner derivation on L_4 is a derivation.

The algebra L_5 . For the element $a \in L_5$ we define the inner derivation on L_5 as follows

$$\begin{aligned} ad_a(x) &= [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (x_1a_2 - x_2a_1)e_1 + (x_3a_4 - x_4a_3)e_3, \end{aligned}$$

where $x \in L_5$. Let Δ be a local inner derivation of the algebra L_5 .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 we obtain the following:

$$\Delta(e_1) = a_{21}e_1, \quad \Delta(e_2) = -a_{12}e_1, \quad \Delta(e_3) = a_{43}e_3, \quad \Delta(e_4) = -a_{34}e_3.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{34}e_3 + a_{43}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_6 . For the element $a \in L_6$ we define the inner derivation on L_6 as follows

$$\begin{aligned} ad_a(x) &= [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (2x_3a_1 - 2x_1a_3)e_1 + (2x_2a_3 - 2x_3a_2)e_2 + (x_1a_2 - x_2a_1)e_3, \end{aligned}$$

where $x \in L_6$. Let Δ be a local inner derivation of the algebra L_6 .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_1 + e_2, e_1 + e_3, e_2 + e_3$ we obtain the following:

$$\begin{aligned} \Delta(e_1) &= -2a_{31}e_1 + a_{21}e_3, \quad \Delta(e_2) = 2a_{32}e_2 - a_{12}e_3, \\ \Delta(e_3) &= 2a_{13}e_1 - 2a_{23}e_2, \quad \Delta(e_4) = 0, \quad \Delta(e_1 + e_2) = -2a_3e_1 + 2a_3e_2 + (a_2 - a_1)e_3, \\ \Delta(e_1 + e_3) &= (2b_1 - 2b_3)e_1 - 2b_2e_2 + b_2e_3, \quad \Delta(e_2 + e_3) = 2c_1e_1 + (2c_3 - 2c_2)e_2 - c_1e_3. \end{aligned}$$

Using the technique of defining equality (3.1'), we will have the following relations:

- $\Delta(e_1 + e_2) = \Delta(e_1) + \Delta(e_2) \Rightarrow a_{32} = a_{31}$;
- $\Delta(e_1 + e_3) = \Delta(e_1) + \Delta(e_3) \Rightarrow a_{23} = a_{21}$;
- $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3) \Rightarrow a_{13} = a_{12}$.

From these obtained equalities we have

$$\Delta(e_1) = -2a_{31}e_1 + a_{21}e_3, \quad \Delta(e_2) = 2a_{31}e_2 - a_{12}e_3, \quad \Delta(e_3) = 2a_{12}e_1 - 2a_{21}e_2, \quad \Delta(e_4) = 0$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3]$. This means that the operator Δ is an inner derivation.

The algebra L_7 . For the element $a \in L_7$ we define the inner derivation on L_7 as follows

$$\begin{aligned} ad_a(x) &= [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (x_1a_2 - x_2a_1)e_3 + (x_1a_3 - x_3a_1)e_4, \end{aligned}$$

where $x \in L_7$. Let Δ be a local inner derivation of the algebra L_7 .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3$ we obtain the following:

$$\begin{aligned} \Delta(e_1) &= a_{21}e_3 + a_{31}e_4, \quad \Delta(e_2) = -a_{12}e_3, \quad \Delta(e_3) = -a_{13}e_4, \quad \Delta(e_4) = 0, \\ \Delta(e_2 + e_3) &= -a_1e_3 - a_1e_4. \end{aligned}$$

From the equality $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3)$ we get $a_{13} = a_{12}$ and

$$\Delta(e_3) = -a_{12}e_4.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3]$. This means that the operator Δ is an inner derivation.

The algebra L_8 . For the element $a \in L_8$ we define the inner derivation on L_8 as follows

$$\begin{aligned} ad_a(x) &= [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (x_1a_2 - x_2a_1)e_2 + (x_1a_3 - x_3a_1)e_3 + (\alpha x_1a_4 - \alpha x_4a_1)e_4, \end{aligned}$$

where $x \in L_8$.

Let Δ be a local inner derivation of the algebra L_8 .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3 + e_4$ we obtain the following:

$$\begin{aligned} \Delta(e_1) &= a_{21}e_2 + a_{31}e_3 + \alpha a_{41}e_4, \quad \Delta(e_2) = -a_{12}e_2, \quad \Delta(e_3) = -a_{13}e_3, \quad \Delta(e_4) = -\alpha a_{14}e_4, \\ \Delta(e_2 + e_3 + e_4) &= -a_1e_2 - a_1e_3 - \alpha a_1e_4. \end{aligned}$$

From the equality $\Delta(e_2 + e_3 + e_4) = \Delta(e_2) + \Delta(e_3) + \Delta(e_4)$ we get $a_{13} = a_{14} = a_{12}$ and

$$\Delta(e_3) = -a_{12}e_3, \quad \Delta(e_4) = -\alpha a_{12}e_4.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_9 . For the element $a \in L_9$ we define the inner derivation on L_9 as follows

$$\begin{aligned} ad_a(x) &= [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (\alpha x_1a_4 - x_4\alpha a_1)e_2 + (x_1a_2 - x_2a_1 - \beta a_4x_1 + \beta a_1x_4)e_3 + (x_1a_3 - x_3a_1 - a_4x_1 + a_1x_4)e_4, \end{aligned}$$

where $x \in L_9$.

Let Δ be a local inner derivation of the algebra L_9 .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3, e_2 + e_4, e_3 + e_4$ we obtain the following:

$$\begin{aligned} \Delta(e_1) &= \alpha a_{41}e_2 + (a_{21} - \beta a_{41})e_3 + (a_{31} - \beta a_{41})e_4, \quad \Delta(e_2) = -a_{12}e_3, \quad \Delta(e_3) = -a_{13}e_4, \\ \Delta(e_4) &= -\alpha a_{14}e_2 + \beta a_{14}e_3 + a_{14}e_4, \quad \Delta(e_2 + e_3) = -a_1e_3 - a_1e_4, \end{aligned}$$

$$\Delta(e_2 + e_4) = -\alpha b_1e_2 + (-b_1 + \beta b_1)e_3 + b_1e_4, \quad \Delta(e_3 + e_4) = -\alpha c_1e_2 + \beta c_1e_3;$$

Then we have the follows

- $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3) \Rightarrow a_{13} = a_{12};$
- $\Delta(e_2 + e_4) = \Delta(e_2) + \Delta(e_4) \Rightarrow a_{14} = a_{12};$
- $\Delta(e_3 + e_4) = \Delta(e_3) + \Delta(e_4) \Rightarrow a_{14} = a_{13}.$

From these obtained equalities we get

$$\Delta(e_3) = -a_{12}e_4, \quad \Delta(e_4) = -\alpha a_{12}e_2 + \beta a_{12}e_3 + a_{12}e_4.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_{10} . For the element $a \in L_{10}$ we define the inner derivation on L_{10} as follows

$$ad_a(x) = [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ = (\alpha x_1 a_4 - x_4 \alpha a_1) e_2 + (x_1 a_2 - x_2 a_1 - \alpha a_4 x_1 - \alpha a_1 x_4) e_3 + (x_1 a_3 - x_3 a_1) e_4,$$

where $x \in L_{10}$.

Let Δ be a local inner derivation of the algebra L_{10} .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3, e_3 + e_4$ we obtain the following:

$$\Delta(e_1) = \alpha a_{41} e_2 + (a_{21} + \alpha a_{41}) e_3 + a_{31} e_4, \Delta(e_2) = -a_{12} e_3, \Delta(e_3) = -a_{13} e_4, \\ \Delta(e_4) = -\alpha a_{14} e_2 - \alpha a_{14} e_3, \Delta(e_2 + e_3) = -a_1 e_3 - a_1 e_4, \Delta(e_3 + e_4) = -\alpha b_1 e_2 - \alpha b_1 e_3 - b_1 e_4.$$

Then we have the follows

- $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3) \Rightarrow a_{13} = a_{12};$
- $\Delta(e_3 + e_4) = \Delta(e_3) + \Delta(e_4) \Rightarrow a_{14} = a_{13};$

and

$$\Delta(e_3) = -a_{12} e_4, \Delta(e_4) = -\alpha a_{12} e_2 - \alpha a_{12} e_3.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_{11} . Repeating the previous technique for this algebra, we get that every local inner derivation on L_{11} is a derivation.

The algebra L_{12} . For the element $a \in L_{12}$ we define the inner derivation on L_{12} as follows

$$ad_a(x) = [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ = \left(\frac{1}{3}x_1a_2 - \frac{1}{3}x_2a_1\right)e_2 + \left(x_1a_2 - x_2a_1 + \frac{1}{3}x_1a_3 - \frac{1}{3}x_3a_1\right)e_3 + \left(\frac{1}{3}x_1a_4 - \frac{1}{3}x_4a_1\right)e_4.$$

where $x \in L_{12}$.

Let Δ be a local inner derivation of the algebra L_{12} .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_3 + e_4, e_2 + e_4$ we obtain the following:

$$\Delta(e_1) = \frac{1}{3}a_{21}e_2 + \left(a_{21} + \frac{1}{3}a_{31}\right)e_3 + \frac{1}{3}a_{41}e_4, \Delta(e_2) = -\frac{1}{3}a_{12}e_2 - a_{12}e_3, \Delta(e_3) = -\frac{1}{3}a_{13}e_3, \\ \Delta(e_4) = -\frac{1}{3}a_{14}e_4, \Delta(e_3 + e_4) = -\frac{1}{3}a_1e_3 - \frac{1}{3}a_1e_4, \Delta(e_2 + e_4) = -\frac{1}{3}b_1e_2 - b_1e_3 - \frac{1}{3}b_1e_4,$$

Then we have the follows

- $\Delta(e_3 + e_4) = \Delta(e_3) + \Delta(e_4) \Rightarrow a_{14} = a_{13};$
- $\Delta(e_2 + e_4) = \Delta(e_2) + \Delta(e_4) \Rightarrow a_{14} = a_{12};$

and

$$\Delta(e_3) = -\frac{1}{3}a_{12}e_3, \Delta(e_4) = -\frac{1}{3}a_{12}e_4.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_{14} . For the element $a \in L_{14}$ we define the inner derivation on L_{14} as follows

$$\begin{aligned} ad_a(x) = [x, a] &= [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (x_1a_3 - x_3a_1)e_2 + (x_1a_2 - x_2a_1)e_3 + (x_2a_3 - x_3a_2)e_4. \end{aligned}$$

where $x \in L_{14}$.

Let Δ be a local inner derivation of the algebra L_{14} .

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_1 + e_3, e_2 + e_3, e_1 + e_2$ we obtain the following:

$$\begin{aligned} \Delta(e_1) &= a_{31}e_2 + a_{21}e_3, \quad \Delta(e_2) = -a_{12}e_3 + a_{32}e_4, \quad \Delta(e_3) = -a_{13}e_2 - a_{23}e_4, \quad \Delta(e_4) = 0, \\ \Delta(e_1 + e_3) &= (a_3 - a_1)e_2 + a_2e_3 - a_2e_4, \quad \Delta(e_2 + e_3) = -b_1e_2 - b_1e_3 + (b_3 - b_2)e_4, \\ \Delta(e_1 + e_2) &= c_3e_2 + (c_2 - c_1)e_3 + c_3e_4. \end{aligned}$$

Then we have the follows

- $\Delta(e_1 + e_3) = \Delta(e_1) + \Delta(e_3) \Rightarrow a_{23} = a_{21};$
- $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3) \Rightarrow a_{13} = a_{12};$
- $\Delta(e_1 + e_2) = \Delta(e_1) + \Delta(e_2) \Rightarrow a_{32} = a_{31};$

and

$$\Delta(e_2) = -a_{12}e_3 + a_{31}e_4, \quad \Delta(e_3) = -a_{12}e_2 - a_{21}e_4.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3]$. This means that the operator Δ is an inner derivation.

On four-dimensional Lie algebras $L_1 - L_{12}$ and L_{14} , an arbitrary local inner derivation is an inner derivation.

The algebras L_{13} and L_{15} admit a local inner derivation that is not an inner derivation.

The algebra L_{13} . For the element $a \in L_{13}$ we define the inner derivation on L_{13} as follows

$$\begin{aligned} [x, a] &= [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] = \\ &= (x_1a_2 - x_2a_1)e_2 + (x_1a_3 - x_3a_1)e_3 + (x_2a_3 - x_3a_2 + 2x_1a_4 - 2x_4a_1)e_4, \end{aligned}$$

where $x \in L_{13}$.

Consider an operator

$$\Delta(x) = (a_{42}x_2 + a_{43}x_3 + a_{44}x_4)e_4.$$

This operator is a local inner derivation, because $\Delta(x) = [x, \varphi(x)]$ is true for the function

$$\varphi(x) = \begin{cases} \frac{1}{x_2}(a_{42}x_2 + a_{43}x_3 + a_{44}x_4)e_3, & x_1 = 0, x_2 \neq 0, \\ -\frac{a_{44}}{2}e_1, & x_1 = x_2 = x_3 = 0, \\ -\frac{1}{x_3}(a_{43}x_3 + a_{44}x_4)e_2, & x_1 = x_2 = 0, x_3 \neq 0, \\ \frac{1}{2x_1}(a_{42}x_2 + a_{43}x_3 + a_{44}x_4)e_4, & x_1 \neq 0. \end{cases}$$

Now let's show that Δ is not an inner derivation. Let $u = e_1 + 2e_2 + e_3 + e_4$ and $v = 2e_1 + e_2 + 2e_3 + 0,5e_4$. From L_{13} multiplications we get $[u, v] = 0$. Then

$$\Delta(u) = (2a_{42} + a_{43} + a_{44})e_3, \Delta(v) = (a_{42} + 2a_{43} + 0,5a_{44})e_3 \text{ and}$$

$$\Delta([u, v]) = 0, \Delta(u)v + u\Delta(v) = -(3a_{42} + 1,5a_{44})e_3 + 3a_{43}e_4 \neq 0.$$

Hence it follows that

$$\Delta([u, v]) \neq \Delta(u)v + u\Delta(v).$$

Since Δ is not a derivation, it is also not an inner derivation.

The algebra L_{15} . As in the case of the algebra L_{13} , the operator

$$\Delta(x) = a_{21}x_1e_2 + a_{31}x_1e_3 = [x, \phi(x)],$$

$$\phi(x) = \begin{cases} 0, & x_1 = 0, \\ (a_{31} + \frac{a_{21}}{\alpha})e_2 - \frac{a_{21}}{\alpha}e_3 + \frac{(\alpha a_{31} + a_{21})x_3 + a_{21}x_2}{\alpha x_1}, & x_1 \neq 0 \end{cases}$$

is a local inner derivation which is not an inner derivation.

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Rezyume. Mazkur maqolada to'rt o'chamli Li algebraLARining local ichki differentsiallashlari o'rganilgan.

Резюме: В этой статье изучены локальные внутренние дифференцирования четырехмерный алгебр Ли.

Kalit so'zlar: Li algebra, differentsiallash, ichki differentsiallash, local ichki differentsiallash.

Ключевые слова. Алгебра Ли, дифференцирование, внутренние дифференцирование, локальные внутренние дифференцирование.