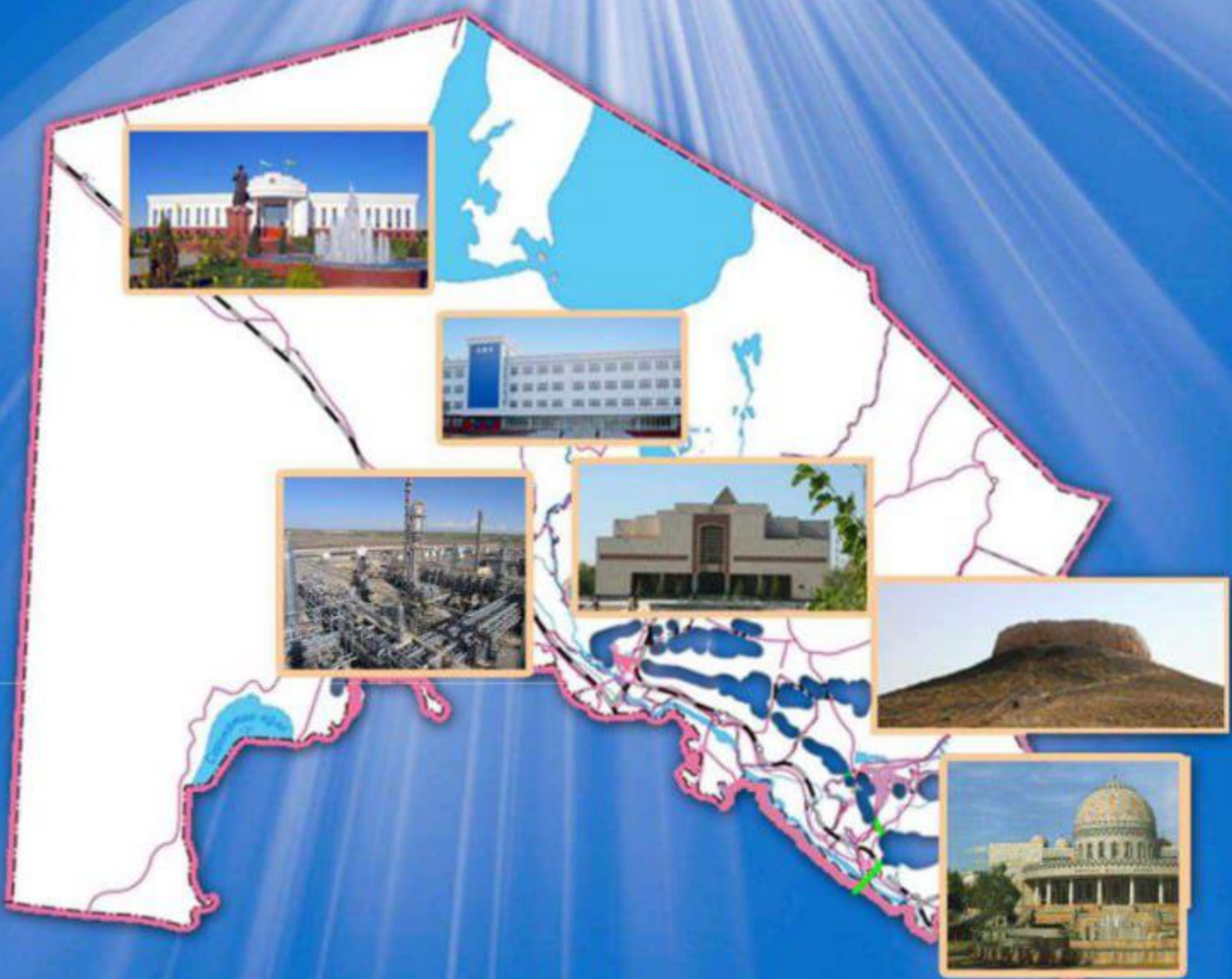
ҚОРАҚАЛПОҒИСТОНДА ФАН ВА ТАЪЛИМ

ҚАРАҚАЛПАҚСТАНДА ИЛИМ ҲӘМ ТӘЛИМ

НАУКА И ОБРАЗОВАНИЕ В КАРАКАЛПАКСТАНЕ

SCIENCE AND EDUCATION IN KARAKALPAKSTAN

1/1-сон
Нукус
2022 й.
ISSN 2181-9203



Science and Education in Karakalpakstan ISSN 2181-9203 №1/1 (22) 2022

Editorial office address:
1 Ch. Abdirov Str., Nukus 230100,
Karakalpakstan, Uzbekistan
Phone: 223-60-19

LOCAL INNER DERIVATIONS ON FOUR-DIMENSIONAL LIE ALGEBRAS

Alauadinov A.K.[1], Jumaeva Ch.I.[2]
[1]*Karakalpak state university,* [2]*V.I.Romanovskiy Institute of Mathematics Uzbekistan Academy of Sciences*

Summary: *In this paper we investigate local inner derivations of four-dimensional Lie algebras.*
Key words: *Lie algebra, derivation, inner derivation, local derivation, local inner derivation*.

1. Introductions. Currently, various types of derivations over algebras are being studied, for example, derivations, local derivations, 2-local derivations, almost inner derivations, etc. Local derivations were first considered in the work of R.Kaidison in 1990 [11] and, independently in the work of D.Larson and A.Surur [12]. In these papers, some conditions are indicated under which local derivations is derivations. R.Kaidison's paper considered local derivations on von Neumann algebras and in some polynomial algebras. L.Molnár [13] introduced the definition of local inner derivations on standard operator algebras.

The first results concern to local and 2-local derivations and automorphisms on finite-dimensional Lie algebras over algebraically closed field of zero characteristic were obtained in [2, 3, 5] and [8]. Namely, in [5] it is proved that every 2-local derivation on a semi-simple Lie algebra L is a derivation and that each finite-dimensional nilpotent Lie algebra with dimension larger than two admits 2-local derivation which is not a derivation. In [2] the authors have proved that every local derivation on a semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. Concerning 2-local automorphism, Z.Chen and D.Wang in [8] prove that if L is a simple Lie algebra of type A_l, D_l or E_k, (k = 6,7,8) over an algebraically closed field of characteristic zero, then every 2-local automorphism of *L* is an automorphism. Finally, in [3] Sh.A.Ayupov and K.Kudaybergenov generalized this result of [8] and proved that every 2-local automorphism of a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is an automorphism. Moreover, they show also that every nilpotent Lie algebra with finite-dimension larger than two admits 2-local automorphisms which is not an automorphism. Local automorphisms of certain finite-dimensional simple Lie and Leibniz algebras are investigated in [4]. Almost inner derivations of Lie algebras were introduced by C.S. Gordon and E.N. Wilson [9] in the study of isospectral deformations of compact manifolds. Almost inner derivations of nilpotent, some solvable Lie algebras and some nilpotent Leibniz algebras were studied in the papers by [6] and [1].

2. Preliminaries. To begin with , recall the definition of Lie algebras.

Definition 2.1. An algebra L over field F is called a *Lie algebra* if its multiplication satisfies the identities:

1) $[x,x]=0$,
2) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$,

for all x,y,z in L.

The product $[x,y]$ is called the bracket of x and y. Identity 2) is called the Jacobi identity.

Let L be a finite-dimensional Lie algebra. For Lie algebra L we consider the following central and derived series:

$$L^1=L,\ L^i=\left[L^{i-1},L\right],\ i\geq 1,$$
$$L^{[1]}=L,\ L^{[k]}=\left[L^{[k-1]},L^{[k-1]}\right],\ k\geq 1.$$

A Lie algebra L is *nilpotent (solvable)* if there exists $m\geq 1$ such that $L^m=0$ ($L^{[m]}=0$).

Definition 2.2. A *derivation* on a Lie algebra L is a linear map $D:L\to L$ which satisfies the Leibniz rule:

$$D([x,y])=[D(x),y]+[x,D(y)] \qquad (2.1)$$

for any $x, y \in L$. The set of all derivations of a Lie algebra denoted by $Der(L)$. Let a be an element of a Lie algebra L. Consider the operator of $ad_a : L \to L$ defined by $ad_a(x) = [x, a]$. This operator is a derivation and called *inner derivation*. The set of all inner derivations of a Lie algebra denoted by $InDer(L)$.

Definition 2.3. A linear operator Δ is called a *local derivation* if for any $x \in L$, there exists a derivation $D_x : L \to L$ (depending on x) such that $\Delta(x) = D_x(x)$.

Definition 2.4. A linear operator Δ is called a *local inner derivation* if for any $x \in L$, there exists a inner derivation $ad_x : L \to L$ (depending on x) such that $\Delta(x) = ad_x(x)$.

We present the following theorem which gives a classification of arbitrary four-dimensional Lie algebras.

Theorem 2.1. [7]. An arbitrary four-dimensional Lie algebra is isomorphic to one of the following algebras: L_0 : abelian;

$$L_1 : [e_1, e_2] = e_3;$$
$$L_2 : [e_1, e_2] = e_1;$$
$$L_3 : [e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3;$$
$$L_4 : [e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3, \lambda \in C^*, |\lambda| \leq 1;$$
$$L_5 : [e_1, e_2] = e_1, [e_3, e_4] = e_3;$$
$$L_6 : [e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2;$$
$$L_7 : [e_1, e_2] = e_3, [e_1, e_3] = e_4;$$
$$L_8 : [e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = \alpha e_4, \alpha \in C^*;$$
$$L_9 : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha e_2 - \beta e_3 - e_4, \alpha \in C^*, \beta \in C;$$
$$L_{10} : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \alpha(e_2 + e_3), \alpha \in C^*;$$
$$L_{11} : [e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_2;$$
$$L_{12} : [e_1, e_2] = \frac{1}{3}e_2 + e_3, [e_1, e_3] = \frac{1}{3}e_3, [e_1, e_4] = \frac{1}{3}e_4;$$
$$L_{13} : [e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = 2e_4, [e_2, e_3] = e_4;$$
$$L_{14} : [e_1, e_2] = e_3, [e_1, e_3] = e_2, [e_2, e_3] = e_4;$$
$$L_{15} : [e_1, e_2] = e_3, [e_1, e_3] = -\alpha e_2 + e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, \alpha \in C;$$

3. Main results. In this section we will consider local inner derivations of four-dimensional Lie algebras.

The following theorem is the main result of this work.

Theorem 3.1. Any local inner derivation on the algebras $L_1 - L_{12}$, L_{14} is an inner derivation, and on the algebras L_{13} and L_{15} there exists a local inner derivation which is not inner derivation.

Proof. We verify that local inner derivations on the algebras $L_1 - L_{12}$, L_{14} are inner derivations.

The algebra L_0. First, consider the algebra L_0 from Theorem 2.1. Inner derivations on the algebra L_0 are zero. Therefore, any local inner derivation is also zero.

The algebra L_1. For the element $a=\sum_{i=1}^{4}a_ie_i\in L_1$ we define the inner derivation on L_1 as follows

$ad_a(x)=[x,a]=[x_1e_1+x_2e_2+x_3e_3+x_4e_4,a_1e_1+a_2e_2+a_3e_3+a_4e_4]=(x_1a_2-x_2a_1)e_3$ where $x=\sum_{i=1}^{4}x_ie_i\in L_1$.

Let Δ be a local inner derivation of the algebra L_1. By definition of local inner derivation, checking the equality $\Delta(e_i)=ad_{e_i}(e_i)$ $(i=1,2,3,4)$ for the basis $\{e_1,e_2,e_3,e_4\}$, we obtain the following:

$$\Delta(e_1)=a_{21}e_3,\ \ \Delta(e_2)=-a_{12}e_3,\ \ \Delta(e_3)=\Delta(e_4)=0.$$

Since the operator Δ is linear, then

$$\Delta(x)=\Delta(x_1e_1+x_2e_2+x_3e_3+x_4e_4)=(a_{21}x_1-a_{12}x_2)e_3.$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2].$ This means that the operator Δ is an inner derivation.

The algebra L_2. Repeating the previous technique for this algebra, we get that every local inner derivation on L_2 is a derivation.

For the algebras L_3 and L_4, we check the equality of $\Delta(x)=ad_x(x)$ for values of x equal to e_1,e_2,e_3,e_4 and e_2+e_3

The algebra L_3. For the element $a=\sum_{i=1}^{4}a_ie_i\in L_3$ we define the inner derivation on L_3 as follows

$$ad_a(x)=[x,a]=[x_1e_1+x_2e_2+x_3e_3+x_4e_4,a_1e_1+a_2e_2+a_3e_3+a_4e_4]=$$
$$=(x_1a_2+x_1a_3-x_2a_1-x_3a_1)e_2+(x_1a_3-x_3a_1)e_3,$$

where $x\in L_3$.

Let Δ be a local inner derivation of the algebra L_3. Checking the equality of $\Delta(x)=ad_x(x)$ for values of x equal to e_1,e_2,e_3,e_4 and e_2+e_3, we obtain the following:

$$\begin{aligned}&\Delta(e_1)=(a_{21}+a_{31})e_2+a_{31}e_3,\\&\Delta(e_2)=-a_{12}e_2,\\&\Delta(e_3)=-a_{13}e_2-a_{13}e_3,\\&\Delta(e_4)=0,\end{aligned}\qquad(3.1)$$

$$\Delta(e_2+e_3)=-2a_1e_2-a_1e_3.$$

From the equality $\Delta(e_2+e_3)=\Delta(e_2)+\Delta(e_3)$ i.e. $-2a_1e_2-a_1e_3=-a_{12}e_2-a_{13}e_2-a_{13}e_3$ we get $a_{13}=a_{12}$. Substituting the resulting equality into (3.1) we have that

$$\Delta(e_3)=-a_{12}e_2-a_{12}e_3.\qquad(3.1')$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2+a_{31}e_3].$ This means that the operator Δ is an inner derivation.

The algebra L_4. Repeating the previous technique for this algebra, we get that every local inner derivation on L_4 is a derivation.

The algebra L_5. For the element $a \in L_5$ we define the inner derivation on L_5 as follows

$$ad_a(x) = [x,a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] =$$
$$= (x_1a_2 - x_2a_1)e_1 + (x_3a_4 - x_4a_3)e_3,$$

where $x \in L_5$.Let Δ be a local inner derivation of the algebra L_5.

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 we obtain the following:

$$\Delta(e_1) = a_{21}e_1, \quad \Delta(e_2) = -a_{12}e_1, \quad \Delta(e_3) = a_{43}e_3, \quad \Delta(e_4) = -a_{34}e_3.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{34}e_3 + a_{43}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_6. For the element $a \in L_6$ we define the inner derivation on L_6 as follows

$$ad_a(x) = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] =$$
$$= (2x_3a_1 - 2x_1a_3)e_1 + (2x_2a_3 - 2x_3a_2)e_2 + (x_1a_2 - x_2a_1)e_3,$$

where $x \in L_6$.Let Δ be a local inner derivation of the algebra L_6.

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_1 + e_2, e_1 + e_3, e_2 + e_3$ we obtain the following:

$$\Delta(e_1) = -2a_{31}e_1 + a_{21}e_3, \quad \Delta(e_2) = 2a_{32}e_2 - a_{12}e_3,$$
$$\Delta(e_3) = 2a_{13}e_1 - 2a_{23}e_2, \quad \Delta(e_4) = 0, \quad \Delta(e_1 + e_2) = -2a_3e_1 + 2a_3e_2 + (a_2 - a_1)e_3,$$
$$\Delta(e_1 + e_3) = (2b_1 - 2b_3)e_1 - 2b_2e_2 + b_2e_3, \quad \Delta(e_2 + e_3) = 2c_1e_1 + (2c_3 - 2c_2)e_2 - c_1e_3.$$

Using the technique of defining equality (3.1'), we will have the following relations:

- $\Delta(e_1 + e_2) = \Delta(e_1) + \Delta(e_2) \Rightarrow a_{32} = a_{31}$;
- $\Delta(e_1 + e_3) = \Delta(e_1) + \Delta(e_3) \Rightarrow a_{23} = a_{21}$;
- $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3) \Rightarrow a_{13} = a_{12}$.

From these obtained equalities we have

$$\Delta(e_1) = -2a_{31}e_1 + a_{21}e_3, \quad \Delta(e_2) = 2a_{31}e_2 - a_{12}e_3, \quad \Delta(e_3) = 2a_{12}e_1 - 2a_{21}e_2, \quad \Delta(e_4) = 0$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3]$. This means that the operator Δ is an inner derivation.

The algebra L_7. For the element $a \in L_7$ we define the inner derivation on L_7 as follows

$$ad_a(x) = [x,a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] =$$
$$= (x_1a_2 - x_2a_1)e_3 + (x_1a_3 - x_3a_1)e_4,$$

where $x \in L_7$.Let Δ be a local inner derivation of the algebra L_7.

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_2 + e_3$ we obtain the following:

$$\Delta(e_1) = a_{21}e_3 + a_{31}e_4, \quad \Delta(e_2) = -a_{12}e_3, \quad \Delta(e_3) = -a_{13}e_4, \quad \Delta(e_4) = 0,$$
$$\Delta(e_2 + e_3) = -a_1e_3 - a_1e_4.$$

From the equality $\Delta(e_2+e_3)=\Delta(e_2)+\Delta(e_3)$ we get $a_{13}=a_{12}$ and

$$\Delta(e_3)=-a_{12}e_4.$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2+a_{31}e_3]$. This means that the operator Δ is an inner derivation.

The algebra L_8. For the element $a\in L_8$ we define the inner derivation on L_8 as follows

$$ad_a(x)=[x,a]=[x_1e_1+x_2e_2+x_3e_3+x_4e_4,a_1e_1+a_2e_2+a_3e_3+a_4e_4]=$$
$$=(x_1a_2-x_2a_1)e_2+(x_1a_3-x_3a_1)e_3+(\alpha x_1a_4-\alpha x_4a_1)e_4,$$

where $x\in L_8$.

Let Δ be a local inner derivation of the algebra L_8.

Checking the equality of $\Delta(x)=ad_x(x)$ for values of x equal to e_1,e_2,e_3,e_4 and $e_2+e_3+e_4$ we obtain the following:

$$\Delta(e_1)=a_{21}e_2+a_{31}e_3+\alpha a_{41}e_4,\ \Delta(e_2)=-a_{12}e_2,\ \Delta(e_3)=-a_{13}e_3,\ \Delta(e_4)=-\alpha a_{14}e_4,$$
$$\Delta(e_2+e_3+e_4)=-a_1e_2-a_1e_3-\alpha a_1e_4.$$

From the equality $\Delta(e_2+e_3+e_4)=\Delta(e_2)+\Delta(e_3)+\Delta(e_4)$ we get $a_{13}=a_{14}=a_{12}$ and

$$\Delta(e_3)=-a_{12}e_3,\ \Delta(e_4)=-\alpha a_{12}e_4.$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2+a_{31}e_3+a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_9. For the element $a\in L_9$ we define the inner derivation on L_9 as follows

$$ad_a(x)=[x,a]=[x_1e_1+x_2e_2+x_3e_3+x_4e_4,a_1e_1+a_2e_2+a_3e_3+a_4e_4]=$$
$$=(\alpha x_1a_4-x_4\alpha a_1)e_2+(x_1a_2-x_2a_1-\beta a_4x_1+\beta a_1x_4)e_3+(x_1a_3-x_3a_1-a_4x_1+a_1x_4)e_4,$$

where $x\in L_9$.

Let Δ be a local inner derivation of the algebra L_9.

Checking the equality of $\Delta(x)=ad_x(x)$ for values of x equal to e_1,e_2,e_3,e_4 and e_2+e_3,e_2+e_4,e_3+e_4 we obtain the following:

$$\Delta(e_1)=\alpha a_{41}e_2+(a_{21}-\beta a_{41})e_3+(a_{31}-\beta a_{41})e_4,\ \Delta(e_2)=-a_{12}e_3,\ \Delta(e_3)=-a_{13}e_4,$$
$$\Delta(e_4)=-\alpha a_{14}e_2+\beta a_{14}e_3+a_{14}e_4,\ \Delta(e_2+e_3)=-a_1e_3-a_1e_4,$$
$$\Delta(e_2+e_4)=-\alpha b_1e_2+(-b_1+\beta b_1)e_3+b_1e_4,\ \Delta(e_3+e_4)=-\alpha c_1e_2+\beta c_1e_3;$$

Then we have the follows

- $\Delta(e_2+e_3)=\Delta(e_2)+\Delta(e_3)\Rightarrow a_{13}=a_{12}$;
- $\Delta(e_2+e_4)=\Delta(e_2)+\Delta(e_4)\Rightarrow a_{14}=a_{12}$;
- $\Delta(e_3+e_4)=\Delta(e_3)+\Delta(e_4)\Rightarrow a_{14}=a_{13}$.

From these obtained equalities we get

$$\Delta(e_3)=-a_{12}e_4,\qquad \Delta(e_4)=-\alpha a_{12}e_2+\beta a_{12}e_3+a_{12}e_4.$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2+a_{31}e_3+a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_{10}. For the element $a\in L_{10}$ we define the inner derivation on L_{10} as follows

$$ad_a(x)=[x,a]=[x_1e_1+x_2e_2+x_3e_3+x_4e_4, a_1e_1+a_2e_2+a_3e_3+a_4e_4]=$$
$$=(\alpha x_1a_4-x_4\alpha a_1)e_2+(x_1a_2-x_2a_1-\alpha a_4x_1-\alpha a_1x_4)e_3+(x_1a_3-x_3a_1)e_4,$$

where $x\in L_{10}$.

Let Δ be a local inner derivation of the algebra L_{10} .

Checking the equality of $\Delta(x)=ad_x(x)$ for values of x equal to e_1,e_2,e_3,e_4 and e_2+e_3,e_3+e_4 we obtain the following:

$$\Delta(e_1)=\alpha a_{41}e_2+(a_{21}+\alpha a_{41})e_3+a_{31}e_4,\ \Delta(e_2)=-a_{12}e_3,\ \Delta(e_3)=-a_{13}e_4,$$
$$\Delta(e_4)=-\alpha a_{14}e_2-\alpha a_{14}e_3,\ \Delta(e_2+e_3)=-a_1e_3-a_1e_4,\ \Delta(e_3+e_4)=-\alpha b_1e_2-\alpha b_1e_3-b_1e_4.$$

Then we have the follows

- $\Delta(e_2+e_3)=\Delta(e_2)+\Delta(e_3)\Rightarrow\ a_{13}=a_{12};$
- $\Delta(e_3+e_4)=\Delta(e_3)+\Delta(e_4)\Rightarrow\ a_{14}=a_{13};$

and

$$\Delta(e_3)=-a_{12}e_4,\ \Delta(e_4)=-\alpha a_{12}e_2-\alpha a_{12}e_3.$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2+a_{31}e_3+a_{41}e_4]$. This means that the operator Δ is an inner derivation.

The algebra L_{11}. Repeating the previous technique for this algebra, we get that every local inner derivation on L_{11} is a derivation.

The algebra L_{12}. For the element $a\in L_{12}$ we define the inner derivation on L_{12} as follows

$$ad_a(x)=[x,a]=[x_1e_1+x_2e_2+x_3e_3+x_4e_4, a_1e_1+a_2e_2+a_3e_3+a_4e_4]=$$
$$=\left(\frac{1}{3}x_1a_2-\frac{1}{3}x_2a_1\right)e_2+\left(x_1a_2-x_2a_1+\frac{1}{3}x_1a_3-\frac{1}{3}x_3a_1\right)e_3+\left(\frac{1}{3}x_1a_4-\frac{1}{3}x_4a_1\right)e_4.$$

where $x\in L_{12}$.

Let Δ be a local inner derivation of the algebra L_{12}.

Checking the equality of $\Delta(x)=ad_x(x)$ for values of x equal to e_1,e_2,e_3,e_4 and e_3+e_4,e_2+e_4 we obtain the following:

$$\Delta(e_1)=\frac{1}{3}a_{21}e_2+\left(a_{21}+\frac{1}{3}a_{31}\right)e_3+\frac{1}{3}a_{41}e_4,\ \Delta(e_2)=-\frac{1}{3}a_{12}e_2-a_{12}e_3,\ \Delta(e_3)=-\frac{1}{3}a_{13}e_3,$$
$$\Delta(e_4)=-\frac{1}{3}a_{14}e_4,\ \Delta(e_3+e_4)=-\frac{1}{3}a_1e_3-\frac{1}{3}a_1e_4,\ \Delta(e_2+e_4)=-\frac{1}{3}b_1e_2-b_1e_3-\frac{1}{3}b_1e_4,$$

Then we have the follows

- $\Delta(e_3+e_4)=\Delta(e_3)+\Delta(e_4)\Rightarrow\ a_{14}=a_{13};$
- $\Delta(e_2+e_4)=\Delta(e_2)+\Delta(e_4)\Rightarrow\ a_{14}=a_{12};$

and

$$\Delta(e_3)=-\frac{1}{3}a_{12}e_3,\quad \Delta(e_4)=-\frac{1}{3}a_{12}e_4.$$

Then $\Delta(x)=[x,a_{12}e_1+a_{21}e_2+a_{31}e_3+a_{41}e_4]$. This means that the operator Δ is an inner derivation .

The algebra L_{14}. For the element $a \in L_{14}$ we define the inner derivation on L_{14} as follows

$$ad_a(x) = [x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] =$$
$$= (x_1a_3 - x_3a_1)e_2 + (x_1a_2 - x_2a_1)e_3 + (x_2a_3 - x_3a_2)e_4.$$

where $x \in L_{14}$.

Let Δ be a local inner derivation of the algebra L_{14}.

Checking the equality of $\Delta(x) = ad_x(x)$ for values of x equal to e_1, e_2, e_3, e_4 and $e_1 + e_3, e_2 + e_3, e_1 + e_2$ we obtain the following:

$$\Delta(e_1) = a_{31}e_2 + a_{21}e_3,\ \Delta(e_2) = -a_{12}e_3 + a_{32}e_4,\ \Delta(e_3) = -a_{13}e_2 - a_{23}e_4,\ \Delta(e_4) = 0,$$
$$\Delta(e_1 + e_3) = (a_3 - a_1)e_2 + a_2e_3 - a_2e_4,\ \Delta(e_2 + e_3) = -b_1e_2 - b_1e_3 + (b_3 - b_2)e_4,$$
$$\Delta(e_1 + e_2) = c_3e_2 + (c_2 - c_1)e_3 + c_3e_4.$$

Then we have the follows

- $\Delta(e_1 + e_3) = \Delta(e_1) + \Delta(e_3) \Rightarrow a_{23} = a_{21}$;
- $\Delta(e_2 + e_3) = \Delta(e_2) + \Delta(e_3) \Rightarrow a_{13} = a_{12}$;
- $\Delta(e_1 + e_2) = \Delta(e_1) + \Delta(e_2) \Rightarrow a_{32} = a_{31}$;

and

$$\Delta(e_2) = -a_{12}e_3 + a_{31}e_4,\ \Delta(e_3) = -a_{12}e_2 - a_{21}e_4.$$

Then $\Delta(x) = [x, a_{12}e_1 + a_{21}e_2 + a_{31}e_3]$. This means that the operator Δ is an inner derivation.

On four-dimensional Lie algebras $L_1 - L_{12}$ and L_{14}, an arbitrary local inner derivation is an inner derivation.

The algebras L_{13} and L_{15} admit a local inner derivation that is not an inner derivation.

The algebra L_{13}. For the element $a \in L_{13}$ we define the inner derivation on L_{13} as follows

$$[x, a] = [x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4, a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4] =$$
$$= (x_1a_2 - x_2a_1)e_2 + (x_1a_3 - x_3a_1)e_3 + (x_2a_3 - x_3a_2 + 2x_1a_4 - 2x_4a_1)e_4,$$

where $x \in L_{13}$.

Consider an operator

$$\Delta(x) = (a_{42}x_2 + a_{43}x_3 + a_{44}x_4)e_4.$$

This operator is a local inner derivation, because $\Delta(x) = [x, \varphi(x)]$ is true for the function

$$\varphi(x) = \begin{cases} \frac{1}{x_2}(a_{42}x_2 + a_{43}x_3 + a_{44}x_4)e_3, & x_1 = 0,\ x_2 \neq 0, \\ -\frac{a_{44}}{2}e_1, & x_1 = x_2 = x_3 = 0, \\ -\frac{1}{x_3}(a_{43}x_3 + a_{44}x_4)e_2, & x_1 = x_2 = 0,\ x_3 \neq 0, \\ \frac{1}{2x_1}(a_{42}x_2 + a_{43}x_3 + a_{44}x_4)e_4, & x_1 \neq 0. \end{cases}$$

Now let's show that Δ is not an inner derivation. Let $u = e_1 + 2e_2 + e_3 + e_4$ and $v = 2e_1 + e_2 + 2e_3 + 0,5e_4$. From L_{13} multiplications we get $[u,v] = 0$. Then

$$\Delta(u) = (2a_{42} + a_{43} + a_{44})e_3,\ \Delta(v) = (a_{42} + 2a_{43} + 0,5a_{44})e_3 \text{ and}$$

$$\Delta([u,v]) = 0,\ \Delta(u)v + u\Delta(v) = -(3a_{42} + 1,5a_{44})e_3 + 3a_{43}e_4 \neq 0.$$

Hence it follows that

$$\Delta([u,v]) \neq \Delta(u)v + u\Delta(v).$$

Since Δ is not a derivation, it is also not an inner derivation.

The algebra L_{15}. As in the case of the algebra L_{13}, the operator

$$\Delta(x) = a_{21}x_1e_2 + a_{31}x_1e_3 = [x, \phi(x)],$$

$$\phi(x) = \begin{cases} 0,\ x_1 = 0, \\ (a_{31} + \dfrac{a_{21}}{\alpha})e_2 - \dfrac{a_{21}}{\alpha}e_3 + \dfrac{(\alpha a_{31} + a_{21})x_3 + a_{21}x_2}{\alpha x_1},\ x_1 \neq 0 \end{cases}$$

is a local inner derivation which is not an inner derivation.

References

1. Adashev J.K., Kurbanbaev T.K., Almost Inner Derivations of some Nilpotent Leibniz Algebras, Journal of Siberian Federal University. Mathematics & Physics 2020, 13(6), 1–13.
2. Ayupov Sh.A., Kudaybergenov K.K., Local derivations on finite-dimensional Lie algebras // Linear Algebra and its Applications. 493 (2016), p. 381–398.
3. Ayupov Sh.A., Kudaybergenov K.K., 2-Local automorphisms on finite-dimensional Lie algebras // Linear Algebra and its Applications. 507 (2016), p. 121–131.
4. Ayupov Sh.A., Kudaybergenov K.K., Local Automorphisms on Finite-Dimensional Lie and Leibniz Algebras // Algebra, Complex Analysis and Pluripotential Theory, USUZCAMP 2017. Springer Proceedings in Mathematics and Statistics. 264 (2017), p. 31–44.
5. Ayupov Sh.A., Kudaybergenov K.K., Rakhimov I.S., 2-Local derivations on finite-dimensional Lie algebras // Linear Algebra and its Applications. 474 (2015), p. 1–11.
6. Burde D., Dekimpe K., Verbeke B., Almost inner derivations of Lie algebras. Journal of Algebra and Its Applications, 17 (2018), no. 11, 26 pages.
7. Burde D., Steinhoff Ch., Classification of orbit closures of 4-dimensional complex Lie algebras. Journal of algebra. 1999.
8. Chen Z., Wang D., 2-Local automorphisms of finite-dimensional simple Lie algebras // Linear Algebra and its Applications. 486 (2015), p. 335–344.
9. Gordon C.S., Wilson E.N., Isospectral deformations of compact solvmanifolds. J. Differential Geom. 19 (1984), no. 1, 214–256.
10. Jacobson N., Lie algebras, Interscience Publishers, Wiley, New York, 1962 .
11. Kadison R.V., Local derivations, J. Algebra, 130 (1990) 494–509.
12. Larson D.R., Sourour A.R., Local derivations and local automorphisms of B(X), Proc. Sympos. Pure Math. 51 (1990) 187–194.
13. Molnár L., Locally inner derivations of standard operator algebras. (English). Mathematica Bohemica, vol. 121 (1996), issue 1, pp. 1-7

Rezyume. *Mazkur maqolada to'rt o'chamli Li algebralarining local ichki differensiallashlari o'rganilgan.*
Резюме: *В этом статье изучены локальные внутренние дифференцирования четырехмерный алгебр Ли.*

Kalit so'zlar: *Li algebrasi, differensiallash, ichki differensiallash, local ichki differensiallash.*
Ключевые слова. *Алгебра Ли, диффернцирование, внутренние дифференцирование, локальные внутренние дифференцирование.*