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The First Type is Some of the Important Types of the Fredholm Integral and Methods for Solving Them

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Abstract: This article reflects on some important types of the Fredholm integral of the first type, which is considered one of the important types of integral equations, methods for calculating them. Specifically about the Schlemilha integral equation, the derivation of the formula that gives its solution has been studied, and some methods for solving integral equations of the same type have been cited.

Keywords: integral equation, integral equation kernel, Fredholm integral Equations, Schlemilha integral equation.

It is known that the solution of a huge number of practical and theoretical issues of urgent importance of many important fields, such as physics, mechanics, technology, chemical industry, leads directly to the solution of integral equations associated with them. A huge number of scientists, such as Fredholm, Volterra, Mersel, carried out effective research in this area and achieved the main results. We directly use the Fredgolm integral Equations in determining computationally specific numbers and specific functions important for operator theory [1].

In the theory of operators, an important enumerated eigenvalues, in determining the eigenfunctions, we look directly at the Fredholm and Volterra integral equations. This article presents one of the important types of integral equation, the first type Fredgolm integral equation, its types, the analysis of the formula of the Schlemilha integral equation determining the induced solution, and the sequence of solving several examples associated with them. First we give a definition of the integral equation. In the article,we represent the space of continuous functions in the field $[a, b]$ through $C_{[a,b]}$ and the space of integrable functions in the field $[a, b]$ through $L_2(a, b)$.

Definition 1. Such an equation is called an integral equation if the unknown function is under an integral signal obtained by the argument of that function.

If the degree of an unknown function in an integral equation is equal to one, such an equation is called a linear integral equation. Let us be given a function (x) , $x \in [a, b]$ and $I = \{\forall(x, t) \in I : a \leq x \leq b, a \leq t \leq b\}$ in the space of $C_{[a,b]} -$ continuous functions $K(x, t)$. The following equation [2]:

$$\varphi(x) = \lambda \int_a^b K(x, t)\varphi(t)dt + f(x) \ \ (1)$$

the second type is called the Fredholm linear integral equation. Where $\lambda \in \mathbb{R}$ is the numerical parameter, $\varphi(x) -$ is the unknown function, $K(x, t) -$ is the kernel of the integral equation, $f(x) -$ is the free term of the integral equation. Integral equations are classified according to several of their properties. When classifying integral equations by type, integral equations in which the unknown function participates only under the integral sign are called integral equations of the first type, and integral equations in which the unknown function participates both under the integral sign and independently are called integral equations of the second type. Integral equations with both limits constant according to the field of detection - Fredholm integral equations, integral equations with one of the limits variable - Volterra integral equations, integral equations with both limits variable - both limits are also integral equations with variables. According to the level of the unknown function being integrated, when the first level of the unknown function is considered - linear integral equations, when two and higher levels are involved - non-linear integral equations; According to the number of variables of the unknown function, it is divided into one-dimensional, two-dimensional, etc. types. In this article, we will analyze the process of deriving the formula for finding the solution of the first type, one-dimensional Fredholm linear integral equations, which is considered one of the important types of linear integral equations, and we will solve some examples of this type [3].

Since the solution of the linear integral equation of the first type directly depends on the interproportionality of the kernel of the integral equation and the free term of the integral equation, some difficulties arise in its solution. In general, linear integral equations of the first type do not always have a solution. The conditions for having a solution of the first type Fredholm linear integral equation were expressed by Picar through his theorem [1].

Let us be given this first type Fredholm line integral equation:

$$f(x) = \int_a^b K(x,t)\varphi(t)dt \ (2)$$

Theorem 1. (Picar's theorem) Fredholm's linear integral equation of first type has a unique solution belonging to the function space $L_2(a,b)$ when the following conditions are met:

1) The kernel of the integral equation is $K(x,t) -$ real symmetric;

2) This series is convergent

$$\sum_{k=1}^{\infty} \lambda_k^2 f_k^2$$

Here, $\lambda_k - \ K(x,t)$ are the eigenvalues of the kernel, and f_k is determined by the following formula:

$$f_k = \int_a^b f(x)\varphi_k(x)dx \ (3)$$

(3) in expression $\varphi_k(x)$ functions are eigenfunctions corresponding to λ_k eigenvalues, respectively;

3) $\{\varphi_k(x)\}$ system of eigenfunctions $[a,b]$ should be a complete and orthonormal system in the section;

When the above conditions are met, the solution of the integral equation (2) will look like this:

$$\varphi(x) = \sum_k \lambda_k f_k \{\varphi_k(x).$$

We can calculate the exact solution at the same time as checking the existence of a solution of the first type Fredholm linear integral equation given by Picard's theorem. One of the special cases of equation (2) is the Schlemilha integral equation of the following form.

$$f(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi(x\sin\theta)d\theta \ (4)$$

Schlemilha's integral equation is a linear integral equation consisting of a complex function of the variable θ with a parameter equal to $\frac{2}{\pi}$ and as a result of his scientific research, the scientist derived the formula for solving this integral equation as follows:

$$\varphi(x) = f(0) + x\int_0^{\frac{\pi}{2}} f'(x\sin\theta)d\theta \ (5)$$

Below we will analyze the process of deriving this formula:

First, we calculate equation (4) when $x = 0$

$$f(0) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi(0)d\theta = \varphi(0) \ \Rightarrow f(0) = \varphi(0) \ (6)$$

In equation (4), we differentiate both sides of the equation by the parameter x

$$f'(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi'(x\sin\theta)\sin\theta d\theta \ (7)$$

We replace x parameter with $x\sin\psi$ parameter

$$f'(x\sin\psi) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi'(x\sin\psi\sin\theta)\sin\theta d\theta \ (8)$$

Multiplying both sides of the equation (7) by the parameter x, we integrate both sides in the field $[0, \frac{\pi}{2}]$ by ψ

$$x\int_0^{\frac{\pi}{2}} f'(x\sin\psi)d\psi = \frac{2x}{\pi}\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}} \varphi'(x\sin\psi\sin\theta)\sin\theta d\theta\right\}d\psi \ (9)$$

We will try to calculate the right side of the resulting equation using integration methods. The left side will be left unchanged for now. First, we switch the limit of integration so that the limit of the integral does not change since the multiple integral is being considered over the same number of intervals.

$$x\int_0^{\frac{\pi}{2}} f'(x\sin\psi)d\psi = \frac{2x}{\pi}\int_0^{\frac{\pi}{2}}\left\{\int_0^{\frac{\pi}{2}} \varphi'(x\sin\psi\sin\theta)\sin\theta d\psi\right\}d\theta \ (10)$$

(9) we make the following substitution in the integral

$$\sin\chi = \sin\psi\sin\theta \ (11)$$

$$\cos\chi\, d\chi = \cos\psi\sin\theta\, d\psi \sin\theta\, d\psi = \frac{\cos\chi}{\cos\psi}\, d\psi$$

We find the value of $\sin\psi\sin\theta$ from the definition (10), put the result in (9) and calculate the result. In this case, when $\psi = 0, \chi = 0$ When $\psi = \frac{\pi}{2}, \chi = \theta$

$$x\int_0^{\frac{\pi}{2}} f'(x\sin\psi)d\psi = \frac{2x}{\pi}\int_0^{\frac{\pi}{2}}\left\{\int_0^{\theta} \frac{\varphi'(x\sin\chi)\cos\chi}{\cos\psi}d\chi\right\}d\theta \ (12)$$

Using notation (10), we express $\cos\psi$ as follows

$$\sin\psi = \frac{\sin\chi}{\sin\theta}, \cos\psi = \frac{\sqrt{\sin^2\theta - \sin^2\chi}}{\sin\theta} = \frac{\sqrt{\cos^2\chi - \cos^2\theta}}{\sin\theta} \quad (13)$$

In equation (11), we change the order of integration once again. Since there is also a variable in the boundary of the integral on the right-hand side of the equation, the boundary of integration also changes and looks like this:

$$\int_0^{\frac{\pi}{2}} f'(x\sin\psi)d\psi = \frac{2x}{\pi}\int_0^{\frac{\pi}{2}}\left\{\int_0^{\theta}\frac{\varphi'(x\sin\chi)\cos\chi\sin\theta}{\sqrt{\cos^2\chi - \cos^2\theta}}d\chi\right\}d\theta =$$

$$= \frac{2x}{\pi}\int_0^{\frac{\pi}{2}}\left\{\int_{\chi}^{\frac{\pi}{2}}\frac{\varphi'(x\sin\chi)\cos\chi\sin\theta}{\sqrt{\cos^2\chi - \cos^2\theta}}d\theta\right\}d\chi =$$

$$= \frac{2x}{\pi}\int_0^{\frac{\pi}{2}}\varphi'(x\sin\chi)\cos\chi\int_{\chi}^{\frac{\pi}{2}}\frac{\sin\theta}{\sqrt{\cos^2\chi - \cos^2\theta}}d\theta = x\int_0^{\frac{\pi}{2}}\varphi'(x\sin\chi)\cos\chi\, d\chi$$

$$x\int_0^{\frac{\pi}{2}} f'(x\sin\psi)d\psi = x\int_0^{\frac{\pi}{2}}\varphi'(x\sin\chi)\cos\chi\, d\chi \quad (14)$$

$$x\int_0^{\frac{\pi}{2}}\varphi'(x\sin\chi)\cos\chi\, d\chi = \int_0^{\frac{\pi}{2}} d\varphi(x\sin\chi) = \varphi(x\sin\chi)|_{\chi=0}^{\chi=\pi/2} = \varphi(x) - \varphi(0)$$

We put the obtained result in equation (12).

$$x\int_0^{\frac{\pi}{2}} f'(x\sin\psi)d\psi = \varphi(x) - \varphi(0)$$

Using the relation (6), we derive the formula for finding the solution of the Schlemilha integral equation

$$\varphi(x) = f(0) + x\int_0^{\frac{\pi}{2}} f'(x\sin\theta)d\theta$$

Example 1. Solve this integral equation.

$$x = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\varphi(x\sin\theta)d\theta$$

Solution. In Example 1, the issue to Beryl is the Schlemilha integral equation,

$$f(x) = x, f'(x) = x, f(0) = 0$$

$$\varphi(x) = x\int_0^{\frac{\pi}{2}} f'(x\sin\theta)d\theta = x\int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2}x$$

Sometimes the integral Equations required to be solved can be brought to the Schlemilha integral equation by the variable substitution method. We look at solving several integral equations for these cases:

$a)\ 1 + x^2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2k}} \varphi\big(x\sin(k\theta)\big)d\theta\ ;\ k \in \mathbb{Z}\{0\}.$

$b)\ 3x^2 + 2x^3 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi(x\cos(\theta))d\theta$

$c)\ x^2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2k}} \varphi(x\cos(k\theta))d\theta, k \in \mathbb{Z}\{0\}$

From the above integral equations given as a sample, we give the solution 3:

$$x^2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2k}} \varphi(x\cos(k\theta))d\theta, k \in \mathbb{Z}\{0\}$$

Solution:

$$x^2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2k}} \varphi(x\cos(k\theta))d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2k}} \varphi\left(x\sin\left(\frac{\pi}{2} - k\theta\right)\right)d\theta$$

We enter a mark as follows,

$$\frac{\pi}{2} - k\theta = t, -kd\theta = dt$$

The integral boundary also changes in this. $t = \frac{\pi}{2}$ when $\theta = 0$; $t = 0$ when $\theta = \frac{\pi}{2k}$.

$$x^2 = \frac{2}{\pi k}\int_0^{\frac{\pi}{2}} \varphi(x\sin t)dt\ ;\ kx^2 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \varphi(x\sin t)dt$$

We brought the integral equation to the Schlemilha integral equation. Now we calculate using the formula.

$$f(x) = kx^2, f'(x) = 2kx$$

$$\varphi(x) = x\int_0^{\frac{\pi}{2}} f'(x\sin\theta)d\theta = x\int_0^{\frac{\pi}{2}} 2kx\sin\theta d\theta = 2k\,x^2, Javob: \varphi(x) = 2kx^2.$$

Through the method analyzed in the article, we can easily calculate some private representatives of the first type Fredholm linear integral equations.

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