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ANALYSIS OF THE 1D FRACTIONAL DIFFUSION EQUATION WITH INITIAL-BOUNDARY PROBLEM

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Abstract. *In this paper, we consider one-dimensional time-fractional diffusion equation is bounded domain. In this work, we have seen the existence of a solution of the 1st initial boundary value problem for the one-dimensional diffusion equation. It is proved that the solution of problem (1)-(5) exists and is unique. First, we give a definition of the classical solution of the direct problem. Then we studied its features. In the process of solving the equation, we used Foure's method, Mittag-Liffler function, Caputo fractional derivatives, Laplace transforms for Caputo fractional derivative. Finally, we directly show that the solution of the problem exists and is unique.*

Keywords: *fractional diffusion equation; Caputo fractional derivative; Fourier method; Mittag-Leffler function.*

Introduction and problem statements. Fractional diffusion equations are more adequate than integer-order models for describing anomalous diffusion phenomena because fractional order derivatives enable the description of memory and hereditary properties of heterogeneous substances [2, 3]. For the last few decades, fractional diffusion equations have attracted great attention not only from mathematicians and engineers but also from many scientists from fields like biology, physics, chemistry and biochemistry, medicine and finance [3-8]. The time fractional diffusion equations arise when replacing the standard time derivative with time fractional derivatives and can be used to describe super diffusion and sub diffusion phenomena [2, 9-11]. Direct problems, i.e. well-posed initial value problems (Cauchy problem), initial boundary value problems for time-fractional diffusion equations, have attracted much more attention in recent years, for instance, on some uniqueness and existence results we refer readers to works [12-19] and on exact solutions of these problems to [20-21]. However, in some practical situations, a part of boundary data, or initial data, or diffusion coefficient, or source term may not be given and we want to find them by additional measurement data which will yield some fractional diffusion inverse problems. Zheng and Wei in [22, 24] solved the Cauchy problems for the time fractional diffusion equations on a strip domain by a Fourier truncation method and a convolution regularization method. In this paper, we consider one-dimensional time-fractional diffusion equation is bounded domain.

Consider the following one-dimensional time-fractional diffusion equation:

$$\partial_t^\alpha u(x,t) - u_{xx}(x,t) = f(x,t), \qquad (x,t) \in \Omega, \tag{1}$$

In the rectangular

$$\Omega := \{(x,t): 0 < x < l, 0 < t \le T\}.$$

Now, we give definition of direct problem for Eq. (1). Find in the domain Ω a function $u(x,t)$ such that

$$u(x,t) \in C(\overline{\Omega}) \cap C_\gamma^{2,\alpha}(\Omega); \tag{2}$$

$$Lu = f(x,t), \quad (x,t) \in \Omega; \tag{3}$$

$$u(x,0) = \varphi(x), \quad 0 < x < l, 0 < t < T; \tag{4}$$

$$u(0,t) = u(l,t) = 0, \quad 0 < t < T; \tag{5}$$

where φ, f are given functions and ∂_t^α stands for Caputo fractional derivative of order $n-1 < \alpha \le n$ (n is positive integer) in the time variable (see [1])

$$\partial_t^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d\tau, & n-1 < \alpha \le n, \\ u^{(n)}(t), & \alpha = n \in \mathbb{N}, \end{cases}$$

and

$$C_\gamma^{\alpha,n} = \{ u(x,t): u(\cdot,t) \in C^{(n)}(0,l), t \in (0,T], \ \partial_t^\alpha u(\cdot,t) \in C_\gamma(0,T], x \in (0,1)\},$$

$$C_\gamma^{\alpha,0}(0,T] = C_\gamma^\alpha(0,T].$$

where $\alpha > 0, n \in \mathbb{N}, 0 \le \gamma < 1$ be such that $\gamma \le \alpha$ (see [1], p.199) and here

$$C_\gamma(0,T] = \{f(t): t^\gamma f(t) \in C(0,T]\}.$$

$$n-1<\alpha\le n;$$

2. Preliminaries

Two parameter Mittag-Leffler (M-L) function.

$$\mathrm{E}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta,z\in\mathbb{C}$ with $Re(\alpha)>0$ – denote the real part of the complex number α.
Laplace Transform Method for Caputo Fractional Derivatives.
The Laplace transform method formula

$$(L^C\partial_t^\alpha y)(s)=s^\alpha(Ly)(s)-\sum_{j=0}^{l-1}d_j s^{\alpha-j-1}\quad(l-1<\alpha\le l, l\in\mathbb{N}),$$

where $d_j=y^{(j)}(0)$, $j=0,\dots l-1$, (see [1], p.312).

Lemma 1. The initial-value problem of functional differential equation for $\alpha\in(0,1)$.

$$\begin{cases}\partial_t^\alpha v(t)+\lambda v(t)=f(t), & 0<t\le T\\ v(0)=a,\end{cases}$$

where ∂_t^α stands for a Caputo fractional derivative operator, λ,a are constants, then there is a explicit solution which is given in the integral form

$$v(t)=a\mathrm{E}_\alpha(-\lambda t^\alpha)+\int_0^t(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)f(\tau)d\tau.$$

and this solution is unique $v(t)\in C_\gamma^\alpha[0,T]$, where $0\le\gamma<\alpha$, $\mathrm{E}_{\alpha,\beta}(\cdot)$ is the two parameter M-L function.

Proof. (See [1], p. 302).

3. Propositions

I. Let $0<\alpha<1$ and $\beta\in\mathbb{R}$ be arbitrary. We suppose that k is such $\frac{\pi\alpha}{2}<k<min\{\pi,\pi\alpha\}$. Then there exists a constant $C=C(\alpha,\beta,k)>0$ such that

$$\left|\mathrm{E}_{\alpha,\beta}(z)\right|\le\frac{C}{1+|z|},\qquad k\le|arg(z)|\le\pi.$$

II. Let $0<\alpha<1$ and $\lambda>0$, then we have

$$\frac{d}{dt}\mathrm{E}_{\alpha,1}(-\lambda t^\alpha)=-\lambda t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(-\lambda t^\alpha),\qquad t>0\,.$$

III. Let $0<\alpha<1$ and $\lambda>0$, then we have

$$\partial_t^\alpha\mathrm{E}_{\alpha,1}(-\lambda t^\alpha)=-\lambda\mathrm{E}_{\alpha,1}(-\lambda t^\alpha),\qquad t>0\,.$$

IV. Let $\alpha>0$, β>0, and $\lambda>0$, then we have

$$\frac{d}{dt}t^{\beta-1}\mathrm{E}_{\alpha,1}(-\lambda t^\alpha)=t^{\beta-2}\mathrm{E}_{\alpha,1}(-\lambda t^\alpha),\qquad t>0.$$

V. For $0<\alpha<1$, $\eta>0$, we have $0\le\mathrm{E}_{\alpha,\alpha}(-\eta)\le\frac{1}{\Gamma(\alpha)}$. Moreover, $\mathrm{E}_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta>0$.

VI. For $0<\alpha\le\beta\le1$ the following hold:

(i) For $\lambda>0, t^{\beta-1}\mathrm{E}_{\alpha,1}(-\lambda t^\alpha)$ is completely monotonic function.

(ii) For $t\in[0,T]$, we have

$$\mathrm{E}_{\alpha,\beta}(-\lambda t^\alpha)<\infty \text{ and } \int_0^t(t-s)^{\alpha-1}\mathrm{E}_{\alpha,\beta}(-\lambda(t-s)^\alpha)ds<\infty.$$

(iii) Furthermore, for $\lambda\in\mathbb{R}^+, t\in(0,T]$

$$\lambda t^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(-\lambda t^\alpha)\le\frac{1}{t}\frac{\lambda t^\alpha}{1+\lambda t^\alpha}<\infty.$$

4. Investigation of direct problem

Using applying the method of separation of variables, we seek a solution of (1)-(5) with the form

$$u(x,t)=X(x)T(t)\tag{6}$$

Moreover, $f(x,t)\equiv0$.

Carrying out a separation of the variables, we obtain the following one dimensional eigen-value problem:

$$u(0,t)=X(0)T(t)=0.\tag{7}$$

$$X(0)=X(l)=0.\tag{8}$$

Where λ is constant of the separation of variables. The boundary condition for $X(x)$ follows from the corresponding conditions for function $u(x,t)$. For example from

$$u(0,t)=T(t)X(0)=0.$$

it follows $X(0)=0$, since $T(t)\ne0$. We have only non-trivial solution.

The solution of equation (7)-(8) have the form

$$X_n(x)=sin\lambda_n x.$$

The eigenvalue

$$\lambda_n=\frac{\pi n}{l},\qquad n\in\mathbb{N}.$$

have corresponding eigenfunction

$$X_n(x) = A_n sin\lambda_n x.$$

where A_n is some constant factor. We choose this so that norm in L_2 of solution X_n with weigh 1 equals unity

$$\|X(x)\|_{L_{2(0,l)}} = \sqrt{A_n \int_0^l X_n^2\,(x)dx}=\sqrt{A_n \int_0^l sin^2\lambda_n\, xdx};$$

$$(X_n(x); X_m(x))_{L_{2(0,l)}} = 1;$$

Hence $A_n = \sqrt{\frac{2}{l}}$. So

$$X_n(x) = \sqrt{\frac{2}{l}} sin\lambda_n x. \qquad (9)$$

We consider the integral

$$T_n(t) = \int_0^l u(x,t)\, X_n(x)dx. \qquad (10)$$

Introduce an auxiliary integral, namely,

$$T_n{}^{\varepsilon}(t) = \int_0^{l-\varepsilon} u(x,t)\, X_n(x)dx,$$

here ε is given sufficiently small positive value.

$$(\partial_t^{\alpha} T_n^{\varepsilon})(t) = \int_{\varepsilon}^{l-\varepsilon} \partial_t^{\alpha} T_n^{\varepsilon}(t) X_n\,(x)dx = \int_{\varepsilon}^{l-\varepsilon} (u_{xx}\,(x,t) + f(x,t)) X_n(x)dx=$$

$$=\int_{\varepsilon}^{l-\varepsilon} u_{xx}(x,t)\, X_n(x)dx + \int_{\varepsilon}^{l-\varepsilon} f(x,t)\, X_n(x)dx = -\lambda T_n^{\varepsilon}(t) + f_n(t), \qquad (11)$$

where

$$f_n(t) = \int_0^l f(x,t) X_n(x)dx.$$

We conclude that

$$\int_{\varepsilon}^{l-\varepsilon} u_{xx}(x,t)\, X_n(x)dx = X_n(x)u_x(x,t)\Big|_{\varepsilon}^{l-\varepsilon} - \int_{\varepsilon}^{l-\varepsilon} u_x(x,t)\, X_n'(x)dx=$$

$$= X_n'(x)u(x,t)\Big|_{\varepsilon}^{l-\varepsilon} + \int_{\varepsilon}^{l-\varepsilon} u(x,t)\, X_n''(x)dx = -\lambda \int_{\varepsilon}^{l-\varepsilon} u(x,t)\, X_n(x)dx = -\lambda T_n{}^{\varepsilon}(t).$$

From formula (11) arrives

$$(\partial_t^{\alpha} T_n)(t) - \lambda T_n(t) = f_n(t). \qquad (12)$$

The initial condition (2) give:

$$T_n(0) = \int_0^l u(x,t)\, X_n(x)dx = \int_0^l \varphi(x) X_n(x)dx = \varphi_n. \qquad (13)$$

According to the lemma 1, the initial-value problem (12)-(13) has a unique solution in $T_n(t) \in C_{\gamma}^{\alpha}[0,T]$ and it is defined by the formula

$$T_n(t) = \varphi_n \mathrm{E}_{\alpha}(-\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} \mathrm{E}_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) f_n(\tau)d\tau. \qquad (14)$$

This means that solution to problem (1)-(5) is unique, because with $\varphi(x) \equiv 0$ and $f(x,t) \equiv 0$ we get identities $\varphi_n \equiv 0, f_n(\tau) \equiv 0$, and then formula (14) implies that $u_n(t) \equiv 0$. In view of formula (10) the letter equality is equivalent to that

$$\int_0^l u(x,t)\, X_n(x)dx = 0.$$

Since the system $X_n(x)$ is complete in the space $L_2(\Omega)$, the function $u(x,t) = 0$ almost everywhere in Ω and with any $t \in [0,T]$. Since in view of condition (2) the function $u(x,t)$ on $\overline{\Omega}$ we conclude that $u_n(x,t) \equiv 0$ on $\overline{\Omega}$. Thus, we have proved the uniqueness of the solution to problem (1)-(5).

Since we have $\int_0^l u(x,t)\, X_n(x)dx = \varphi_n$ equal to, we bring it to equation (14)

$$T_n(t) = \sqrt{\frac{2}{l}} \int_0^l \varphi(\xi)\, sin\lambda\xi \mathrm{E}_{\alpha}(-\lambda t^{\alpha})d\xi +$$

$$+\sqrt{\frac{2}{l}} \int_0^t \int_0^l f(\xi,\tau) sin\lambda\xi (t-\tau)^{\alpha-1}\, \mathrm{E}_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha})d\xi d\tau.$$

The value of problem (1)-(5) will be equal to the following:

$$u(x,t) = \sum_{n=1}^{\infty} \varphi_n \mathrm{E}_{\alpha}(-\lambda t^{\alpha})\, sin\frac{\pi n}{l} x +$$

$$+\sum_{n=1}^{\infty} \int_0^t (t-\tau)^{\alpha-1} \mathrm{E}_{\alpha,\alpha}(-\lambda(t-\tau)^{\alpha}) f_n(\tau)d\tau\, sin\frac{\pi n}{l} x \qquad (15)$$

5. The existence of a solution

Under certain requirements to functions $f(x,t)$ and $\varphi(x)$ we can prove that the function

$$u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n \mathrm{E}_\alpha(-\lambda t^\alpha)+\int_0^t (t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)f_n(\tau)d\tau\right)\times \sin\frac{\pi n}{l}x$$

is a solution to problem (1)-(5).

Formally termwise differentiating the series in formula (14), we get the following series in formula (15), we get the following series

$$u_{xx}(x,t)=-\left(\frac{\pi}{l}\right)^2\sum_{n=1}^{\infty} n^2 T_n(t)\sin\frac{\pi n}{l}x, \qquad t>0, \tag{16}$$

$$\partial_t^\alpha u(x,t)=\sum_{n=1}^{\infty}\partial_t^\alpha T_n(t)\sin\frac{\pi n}{l}x,\ t>0, \tag{17}$$

Lemma 2. *The following estimates are valid with large n:*

$$|u(x,t)|\le C_1\sum_{n=1}^{\infty}(|\varphi_n|+\|f_n\|),\ t\in[0,T], \tag{18}$$

$$|u_{xx}(x,t)|\le C_2\sum_{n=1}^{\infty} n^2(|\varphi_n|+\|f_n\|),\ t\in[0,T], \tag{19}$$

$$|\partial_t^\alpha u(x,t)|\le C_3\sum_{n=1}^{\infty} n(|\varphi_n|+\|f_n\|),\ t\in[\varepsilon,T], \tag{20}$$

here in after $C_i=max\left\{1,\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}T^{\alpha-\gamma}\right\}$ *are positive constant values independent of* $\varphi(x)$ *and* $f(x,t)$*, and* ε *is a positive sufficiently small value.*

The weighted space $C_\gamma[0,T]$. Then we have,

$$|u(x,t)|\le C_1\sum_{n=1}^{\infty}\left(|\varphi_n|+\frac{\|f_n\|_\gamma}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{-\gamma}d\tau\right)\le$$

$$\le\sum_{n=1}^{\infty}\left(|\varphi_n|+\frac{\|f_n\|_\gamma}{\Gamma(\alpha)}t^{\alpha-\gamma}\frac{\Gamma(\alpha)\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\right)\le\sum_{n=1}^{\infty}\left(|\varphi_n|+T^{\alpha-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\|f_n\|_\gamma\right)$$

$$\le C_1\sum_{n=1}^{\infty}(|\varphi_n|+\|f_n\|); \tag{18*}$$

Now we get two times derivative by space variable in $u(x,t)$, then we get

$$u_{xx}(x,t)=-\left(\frac{\pi}{l}\right)^2\sum_{n=1}^{\infty}n^2\left(\varphi_n\mathrm{E}_\alpha(-\lambda t^\alpha)+\int_0^t(t-\tau)^{\alpha-1}\mathrm{E}_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha)f_n(\tau)d\tau\right)\sin\frac{\pi n}{l}x.$$

Similarly, we estimate $u_{xx}(x,t)$ on Ω:

$$|u_{xx}(x,t)|\le-\left(\frac{\pi}{l}\right)^2\sum_{n=1}^{\infty}n^2\left(|\varphi_n|+\frac{\|f_n\|_\gamma}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{-\gamma}d\tau\right)\le$$

$$\le\sum_{n=1}^{\infty}n^2\left(|\varphi_n|+\frac{\|f_n\|_\gamma}{\Gamma(\alpha)}t^{\alpha-\gamma}\frac{\Gamma(\alpha)\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\right)\le$$

$$\le\sum_{n=1}^{\infty}n^2\left(|\varphi_n|+T^{\alpha-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\|f_n\|_\gamma\right)\le C_2\sum_{n=1}^{\propto}n^2(|\varphi_n|+\|f_n\|); \tag{19*}$$

In estimating (20), we first use Eq. (12), and the estimate (18), then we have

$$|\partial_t^\alpha u(x,t)|\le\left|\sum_{n=1}^{\infty}\partial_t^\alpha T_n(t)\sin\frac{\pi n}{l}x\right|\le\sum_{n=1}^{\infty}(\lambda_n|T_n(t)|+|f_n(t)|)\le$$

$$\le\sum_{n=1}^{\infty}\left(\lambda_n|\varphi_n|+\lambda_n\|f_n(t)\|_\gamma+\|f_n(t)\|\right)\le C_3\sum_{n=1}^{\infty}n(|\varphi_n|+\|f_n\|); \tag{20*}$$

Lemma 3. *If* $\varphi(x)\in C^2[0,l],\ \varphi'''(x)\in L_2(0,l),$ $f(x,t)\in C_{x,t}^{2,0}(\bar{\Omega}),\ f_{xxx}(x,t)\in L_2(\bar{\Omega}),$ *and*

$$\varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=0, \tag{21}$$

$$f(0,t)=f(l,t)=f_{xx}(0,t)=f_{xxx}(l,t)=0,\qquad 0<t\le T, \tag{22}$$

then the following representation are valid:

$$\varphi_n=-\frac{\varphi_n^{(3)}}{\lambda_n^3},\quad f_n=-\frac{f_n^{(3)}(t)}{\lambda_n^3}\ ; \tag{23}$$

here $\varphi_n^{(3)}$, $f_n^{(3)}(t)$ *are coefficients of the expansion of functions* $\varphi'''(x)$ *and* $f_{xxx}(x,t)$ *in series with respect to the function system* $\left\{\sqrt{\frac{2}{l}}\cos\lambda_n x\right\}_{n=1}^{\infty}$ *such that*

$$\sum_{n=1}^{\infty}\left|\varphi_n^{(3)}\right|^2\le\|\varphi'''\|_{L_2[0,l]}, \tag{24}$$

$$\sum_{n=1}^{\infty}\left|f_n^{(3)}(t)\right|^2\le\|f'''\|_{L_2[0,l]\times C[0,T]}. \tag{25}$$

Proof. Indeed, the derivatives of the function up to the second order should be continuous, the third piecewise continuous and besides (21).

$$\varphi_n = \sqrt{\frac{2}{l}}\int_0^l \varphi(x) sin\frac{\pi n}{l}xdx = \sqrt{\frac{2}{l}}\left(-\frac{l}{\pi n}\right)\int_0^l \varphi(x)d\left(cos\frac{\pi n}{l}x\right) =$$

$$= -\frac{\sqrt{2l}}{\pi n}\left[\varphi(x)sin\frac{\pi n}{l}x\Big|_0^l - \int_0^l \varphi'(x)cos\frac{\pi n}{l}xdx\right] = \frac{\sqrt{2l}}{\pi n}\int_0^l \varphi'(x)cos\frac{\pi n}{l}xdx = \frac{l}{\pi n}\varphi'_n; \qquad (26)$$

According to the above calculations, the following equality is also valid for $f_n(t)$

$$f_n(t) = \sqrt{\frac{2}{l}}\int_0^l f'(x,t)cos\frac{\pi n}{l}xdx = \frac{l}{\pi n}f'_n(t); \qquad (27)$$

Now we show the representations (23). Henceforth,

$$\varphi_n = \sqrt{\frac{2}{l}}\int_0^l \varphi(x) sin\frac{\pi n}{l}xdx = \sqrt{\frac{2}{l}}\left(-\frac{l}{\pi n}\right)\int_0^l \varphi(x)d\left(cos\frac{\pi n}{l}x\right) =$$

$$= -\frac{\sqrt{2l}}{\pi n}\left[\varphi(x)sin\frac{\pi n}{l}x\Big|_0^l - \int_0^l \varphi'(x)cos\frac{\pi n}{l}xdx\right] = \frac{\sqrt{2l}}{\pi n}\int_0^l \varphi'(x)cos\frac{\pi n}{l}xdx =$$

$$= \frac{\sqrt{2l}}{\pi n}\frac{l}{\pi n}\int_0^l \varphi'(x)d\left(sin\frac{\pi n}{l}x\right) = \frac{l\sqrt{2l}}{(\pi n)^2}\left[\varphi'(x)sin\frac{\pi n}{l}x\Big|_0^l - \int_0^l \varphi''(x)cos\frac{\pi n}{l}xdx\right] =$$

$$= -\frac{l\sqrt{2l}}{(\pi n)^2}\int_0^l \varphi''(x)sin\frac{\pi n}{l}xdx = \frac{l\sqrt{2l}}{(\pi n)^2}\frac{l}{\pi n}\int_0^l \varphi''(x)d\left(cos\frac{\pi n}{l}x\right) =$$

$$= \sqrt{\frac{2}{l}}\left(\frac{l}{\pi n}\right)^3\left[\varphi''(x)cos\frac{\pi n}{l}x\Big|_0^l - \int_0^l \varphi'''(x)cos\frac{\pi n}{l}xdx\right] =$$

$$= \sqrt{\frac{2}{l}}\left(\frac{l}{\pi n}\right)^3\int_0^l \varphi^{(3)}(x)cos\frac{\pi n}{l}xdx = \left(\frac{l}{\pi n}\right)^3\varphi_n{}^{(3)}; \qquad (28)$$

According to the above calculations, the following equality is also valid for $f_n(t)$:

$$\sqrt{\frac{2}{l}}\int_0^l f_x{}^{(3)}(x,t)cos\frac{\pi n}{l}xdx = \left(\frac{l}{\pi n}\right)^3 f_n{}^{(3)}(t). \qquad (29)$$

The inequality (24) and (25) are obtained by using Bessel inequality.
In view of lemmas 3 with $t \geq \varepsilon > 0$ the series (15),(16),(17) are majorized by the convergent series

$$c\sum_{n=1}^{\infty}\left(\frac{\left|\varphi_n^{(3)}\right|}{n} + \frac{\left\|f_n^{(3)}\right\|}{n}\right).$$

The series (15), (16) and (17) convergent absolutely and uniformly on $\overline{\Omega}$, i.e., function (15) satisfies conditions (1) and (2).
The given assertion implies the next theorem.

Theorem. If $\varphi(x)$ and $f(x,t)$ are functions satisfy conditions of Lemma 3, then the problem (1)-(5) has a unique solution, which is represents the sum of series (15).

Conclusion. In this work, we have seen the existence of a solution of the 1st initial boundary value problem for the one-dimensional diffusion equation. It is proved that the solution of problem (1)-(5) exists and is unique.

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